1. Derivatives

Let *f* be a function whose domain of definition contains a neighborhood of a point z_0. The *derivative* of *f* at z_0, written $f'(z_0)$, is defined by the equation

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \qquad (13)$$

provided this limit exist.

By expressing the variable z in definition (13) in terms of the new complex variable $\Delta z = z - z_0$, we can write that definition as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \qquad (14)$$

Note that, because *f* is defined throughout a neighborhood of z_0, the number $f(z_0 + \Delta z)$ is always defined for $|\Delta z|$ sufficiently small. Let $\Delta w = f(z_0 + \Delta z) - f(z_0)$, then if we write $\frac{dw}{dz}$ for $f'(z)$, equation (14) becomes

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}. \qquad (15)$$

Example

Suppose that $f(z) = z^3$. At any point *z*,

$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^3 - z^3}{\Delta z} = \lim_{\Delta z \to 0} 3z^2 + 3z(\Delta z)^2 + (\Delta z)^2 = 3z^2$. Hence $\frac{dw}{dz} = 3z^2$ or $f'(z) = 3z^2$.

2. Differentiation Formulas

Let *c* be a complex constant, and let *f* be a function whose derivative exist at a point *z*. It is easy to show that

$$\frac{d}{dz}c=0, \qquad \frac{d}{dz}z=1, \qquad \frac{d}{dz}cf(z)=cf'(z). \tag{16}$$

Also, if *n* is a positive integer

$$\frac{d}{dz}z^n=nz^{n-1} \tag{17}$$

Theorem

If the derivation of two function f and g exist at a point z , then

(1) $\frac{d}{dz}\left[f(z)+g(z)\right]=f'(z)+g'(z)$

(2) $\frac{d}{dz}\left[f(z)g(z)\right]=f'(z)g'(z)$

(3) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right]=\frac{f'(z)g(z)-g'(z)f(z)}{\left(g(z)\right)^2}$*, when* $g(z)\neq 0$.

Example

To find the derivative of $\left(2z^2+i\right)^5$. According to the theorem, we have $\frac{d\left(2z^2+i\right)^5}{dz}=5\left(2z^2+i\right)^4(4z)=20z\left(2z^2+i\right)^4$.

Exercises

1. Apply definition (15) of derivative to find $f'(z)$ when

    a. $f(z)=\frac{1}{z},\ z\neq 0$

    b. b. $f(z)=3z^2-2z$

    c. c. $f(z)=\left(z+\frac{1}{2}\right)^4$

2. Apply definition (15) of derivative to find $f'(z)$ when $f(z)=z^3-4z$ at point

    a. $z=z_0$ b. $z=i$

3. Use result in Sec. 2 to find $f'(z)$ when

    a. $f(z)=3z^2-4z+1$

b. $f(z)=\left(1-3z^3\right)^2$

c. $f(z)=\dfrac{z+i}{3z-2}$

4. If the derivation of two function f and g exist at a point z, proof that $\dfrac{d}{dz}\left[f(z)+g(z)\right]=f'(z)+g'(z)$.

5. Show that $f'(z)$ does not exist at any point z when $f(z)=\overline{z}$.