PICARD ITERATION TO SOLVE LINIER AND NONLINIER IVP PROBLEM

Fitriana Yuli Saptaningtyas,M.Si.
anamathuny@gmail.com
Department of Mathematics Educational
Yogyakarta State University

Abstract: *In this paper, we will consider the initial value problem (IVP) using Picard Method. In order to solve the IVP problem using picard method must know the existence and uniqueness about the solution. To determine about the solution of IVP, existence theorem and uniqueness theorem can be used. Then, to describe about uniqueness, existence, and convergence about solution using picard method, Picard Lindelof Theorem can be applied. The paper shows about existence theorem and uniqueness theorem in picard iteration method. The paper also shows implementation picard method to solve non linier and linier IVP which be simulated using MATLAB program.*

1. Introduction

Completion of a problem with the initial value differential equations with analytic methods frequently encountered obstacles. Sometimes it is difficult to find a method of analytic first exact solution to the problem of non-linear differential problems with the initial value.

$$x\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - y = 2 \qquad (1.1)$$

For example, with y (0) = 1. This problem is difficult to solve analytically, so that numerical methods can be used as an alternative solution. Using a numerical method is a method of approach is not always guaranteed the existence of analytic solutions approaching completion. Sometimes numeric settlement also experienced constraints in terms of the convergence of the solution. Various methods can be used to approach the problem accomplishing the initial value of the differential equotion both linear and non-linear. At the completion of the first-order differential equations can be used Eulerian method, Runge Kutta method, newton method, and picard iteration method.

Picard iteration method has been widely applied to solve the initial value problem, one of which has been used to solve the equations of groundwater. From the research has been done by steffen mehl (2006) obtained that the Results show that Picard iterations can be a simple and effective method for the solution of nonlinear, saturated ground water flow problems. In this paper will be described about existence and uniqueness theorem in IVP. This paper also shows about existence and uniqueness theorem in IVP using picard iteration method. Implementation in non linier and linier IVP can be simulated using MATLAB.

2. The Existence And Uniqueness Initial Value Problem Using Picard Iteration Method

In this section, will be described about theorem on initial value problem using picard iteration. The initial value problem $y' - f(x,y), y(0) - y_0$ (2.1)

Theorem 2.1 (exsistence theorem). If $f(x,y)$ are continuous in every point on the rectangle $R = \{(x,y), |x - x0| < a, |y - y0| < b\}$ and bounded in R , $|f(x,y)| \leq K$ in every point (x,y) on R, then there exist minimum one solution on Initial Value Problem in every x on $|x - x_0| < \alpha|$ which α is $minimum\ \{a, b/K\}$.

From Theorem 2.1 if *f(x,y)* are continuous in every point on the rectangle R and bounded in R, then IVP have minimum one solution.

Theorem 2.2 (uniqueness theorem)

$If\ f(x,y)\ and\ \frac{\partial f}{\partial y}\ continu\ in\ every\ point\ (x,y)\ on\ rectangle\ R\ and\ bounded,$

$|f| \leq K, |\frac{\partial f}{\partial y}| \leq M$ in every (x,y) on R, then Initial Value Problem only have one solution y(x).

From theorem 2.2. if *f(x,y)* and $\frac{\partial f}{\partial y}$ are continuous in every point on the rectangle R and bounded in R, then IVP have only one solution. If $\frac{\partial f}{\partial y}$ continuous in every point in rectangle R cause only have one solution. Next theorem, wiil be described about existence and uniqueness about picard iteration method.

Theorem 2.3 (Approximations-Picard Iteration) The solution of initial value problem is found by consturucting recursively a sequence $\{Y_n(x)\}_{n=0}^{\infty}$ of functions

$$y_0(x) = y_{0,}\ and\ y_{n+1}(x) = y_0 + \int_{x_0}^{x} f\big(t, y_n(t)\big)dt\ for\ n \geq 0 \qquad (2.2)$$

then the solution y(x) is given by the limit $y_x = \lim_{n\to\infty} y_n(x)$

Local existence and uniqueness theorem for the Initial Value Problem using Picard Iteration method will be desribed in next theorem.

Definition. 2.1. Let $J = [X_0, X_1]$ be an interval on the real line. We say $f: J \to R$ is Lipschitz Continuous on J if there exsists K>0 such that $\big||f(x) - f(y)|\big| \leq K\big||x - y|\big| for\ x, y\ \in J$

Corollary2.1. Let $J = [X_0, X_1]$ by an interval on the real line and $f: J \to R$. Suppose $f'(x)$ is continuously differentiable on J $(f \in C^1[X_0, X_1])$ then f is Lipschitz continuous on J.

Theorem 2.3. Picard –Lindelof Theorem.

Let $I = [x_-, x_+]$ is an interval in x with $x_- \leq x \leq x_+$ and $J = [y_-, y_+]$ is an interval in y with $y_- \leq y \leq y_+$. If $f: IxJ \to R$ is continuous on its domain and Lipschitz continuous on J, then there exsists $\epsilon > 0$ such that the initial value problem $y' = f(x,y), x \in [x_0, x_0 + \varepsilon], y(x_0) = y_0$ has a unique solution $y \in C'[x_0, x_0 + \varepsilon]]$

From theorem 2.3 if *f* is continuous and lipschitz continuous then the IVP problem has a unique solution. This theorem about local condition in the IVP.

Proof of local unique solution of $y_0(x) = y_{0,}\ and\ y_{n+1}(x) = y_0 + \int_{x_0}^{x} f\big(t, y_n(t)\big)dt\ for\ n \geq 0$

The fact that $f(x,y)$ is continuous and the domain is compact. We can determine maximum slope of the function. If $M = \max_{(x,y)\in IxJ} |f(x,y)|$, and $\delta = \min\ [x_{+,} \frac{y_+ - y_0}{M}, \frac{y_0 - y_-}{M}]$ then if $x \in (0, \delta)$

$$
\begin{aligned}
|y(x) - y_0| &= || \int_0^x f(t, y(t))dt\ || \\
&\leq || \int_0^{\delta} f\big(t, y(t)\big)dt|| \\
&\leq \int_0^{\delta} |f(t, y(t))dt| \\
&\leq M\delta
\end{aligned}
\tag{2.3}
$$

Clearly that $\delta \leq x_+$ and $y_- \leq y_0 - M\delta \leq y(t) \leq y_0 + M\delta \leq y_+$. We can conclude that if the solution exsist for $x \in [0, \delta]$ then $u(x) \in J$ so the solution using picard iteration is exsist.

Convergence of picard iteration method will be discussed above. From the definition of picard iteration method, we know that

$$
\begin{aligned}
\varphi_{n+1} &= y_{n+1} - y_n \\
&= \left[y_0 + \int_0^x f\big(t, y_n(t)\big)dt\right] - [y_0 + \int_0^x f(t, y_{n-1}(t))dt] \\
&= \int_0^t f\big(t, y_n(t)\big) - f\big(t, y_{n-1}(t)\big)dt
\end{aligned}
$$

Taking the absolute value of each side and applying Lipschitz Condition yields

$$
\begin{aligned}
|\varphi_{n+1}| &= |\int_0^t f\big(t, y_n(t)\big) - f\big(t, y_{n-1}(t)\big)dt| \\
&\leq \int_0^x |f\big(t, y_n(t)\big) - f\big(t, y_{n-1}(t)\big)|dt \\
&\leq \int_0^x K|y_n - y_{n-1}| \\
&= K\int_0^x \varphi_n dt
\end{aligned}
$$

So that $||\varphi||_\infty = \max_{x\in[a,b]} |\varphi(x)|$, then we can find condition that

$$||\varphi_{n+1}||_\infty = \max_{x\in[0,\delta]} |\varphi_{n+1}|,$$

$$\leq \max_{x\in[0,\delta]} K|\int_0^x |\varphi_n| dt|$$

$$\leq K\ ||\varphi_n||_\infty$$

The proof resembles the proof of ratio test

$$||\varphi_{n+1}||_\infty \leq K \in ||\varphi_n||_\infty \leq K \in^2 ||\varphi_{n-1}||_\infty \leq \cdots \leq K \in^n ||\varphi_1||_\infty$$

$$|\sum_{k=1}^{M} \varphi_k| \leq \sum_{k=1}^{M} ||\varphi_k||_\infty$$

$$\leq \sum_{k=1}^{M} (Ke)^k ||\varphi_1||_\infty$$

$$= \frac{1-K\epsilon^{M+1}}{1-K\epsilon} ||\varphi_1||_\infty, \text{ Choose } \epsilon < \frac{1}{k}, we\ see\ that\ as\ M \to \infty$$

$$\sum_{k=1}^{\infty} ||\varphi_k||_\infty \leq \frac{1}{1-K\in} ||\varphi_1||_\infty$$

So the sum converges absolutely, then picard iteration converges to $y_\infty = y_0 + \lim_{n\to\infty} \sum_{k=1}^{n} \varphi_k$

The Global existence about solution IVP using Picard Iteration method will be described in next theorem.

Theorem 2.4. Picard –Lindelof Theorem. Let $I = [x_-, x_+]$ is an interval in x with $x_- \leq x \leq x_+$ and If $f: IxR \to R$ is continuous on its domain and globally Lipschitz continuous in second variable. Then, the initial value problem $y' = f(x,y), x \in [x_0, x_0 + \varepsilon], y(x_0) = y_0$ has a unique solution $y \in C'\ for\ x \in I$.

From theorem 2.4 if f is continuous and globallly lipschitz continuous then the IVP problem has a unique solution. This theorem abaout global condition in the IVP.

3. The Aplication of Picard Iteration to solve Linier and Non Linier IVP

In the previous section, we have discussed about convergence and unique of solving IVP using Picard Iteration. In this section, we will use Picard Iteration to solve some linier and non linier IVP problem.

First Case is to Linier IVP Problem. $y' = 4y, y(0) = 1$ (3.1)

Solution Using Picard Iteration $y_n = 1 + 4\int_0^x y_{n-1}(t)dt$, whit $y_0 = 1$

, $y_1 = 1 + 4x$, $y_2 = 1 + 4x + \frac{4x^2}{2!}$, Whit analitic solution is $y = e^{4x}$

The simulation of the solution will be show in Figure 3.1.

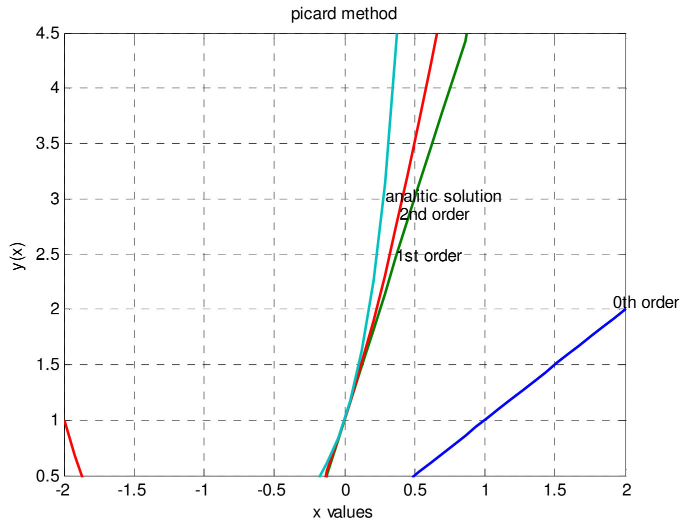

Figure 3.1. Linier IVP Solution Using Picard Method

From this (figure 3.1), will be described that solution using picard method convergence to the analytic solution. The curve of second order solution follow the analytical solution.

Second case is to non linier IVP Problem $y' = xy + 2x - x^3, y(0) = 0$ (3.2)

Solution Using Picard Iteration $y_0 = 0, y_1 = x^2 - \frac{1}{4}x^4, y_2 = x^2 - \frac{1}{4.6}x^4$, whit analytic solution is $y = x^2$. The simulation of the solution will be show in Figure 3.2.

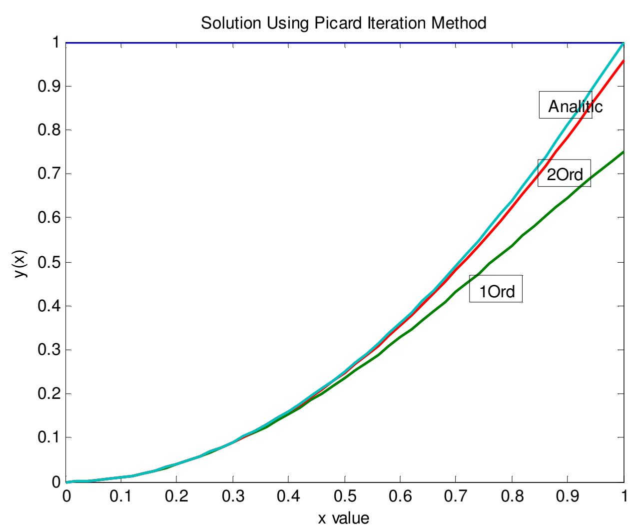

Figure 3.2. Non Linier IVP Solution Using Picard Method

From this (figure 3.2), will be described that solution using picard method convergence to the analytic solution. The curve of second order solution follow the analytical solution. Picard iteration method is easy to used and applied. Before use piacrd iteration method to solve linier and non linier IVP, the existence and uniqueness must be analysis. Picard Lindeloft Theorem can be applied. If $f: IxR \rightarrow R$ is continuous on its domain and globally Lipschitz continuous in second variable. Then, the initial value problem $y' = f(x,y), x \in [x_0, x_0 + \varepsilon], y(x_0) = y_0$ has a unique solution $y \in C'\ for\ x \in I$.

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