0.1. Joint Transformation

The preceding theorems can be extended to apply to functions of several random variables.

THEOREM 1. *If X is a vector of discrete random variables with joint pdf $f_X(x)$ and $Y = g(X)$* defines a one-to-one transformation, then the joint pdf of Y is

$$f_Y(y_1, y_2, ..., y_k) = f_X(x_1, x_2, ..., x_k)$$

where $x_1, x_2, ..., x_k$ are the solutions of $y = g(x)$ and consequently depend on $y_1, y_2, ..., y_k$.

If the transformation is not one-to-one, and if a partition exists, say $A_1, A_2, ...$, such that the equation $y = g(x)$ has a unique solution $x = x_j$ or $x_j = x_{1j}, x_{2j}, ..., x_{kj}$ over A_j, then the pdf of Y is

$$f_Y(y_1, y_2, ..., y_k) = \sum_j f_X(x_{1j}, x_{2j}, ..., x_{kj})$$

THEOREM 2. *Suppose that $X = X_1, X_2, ..., X_k$ is a vector of continuous random variables with joint pdf $f_X(x_1, x_2, ..., x_k) > 0$ on A, and $Y = Y_1, Y_2, ..., Y_k$ is defined by the one-to-one transformation*

$$Y_i = g_i(X_1, X_2, ..., X_k), i = 1, 2, ..., k$$

If the Jacobian is continuous and nonzero over the range of the transformation, then the joint pdf of Y is

$$f_Y(y_1, y_2, ..., y_k) = f_X(x_1, x_2, ..., x_k) |J|$$

where $x = (x_1, x_2, ..., x_k)$ is the solution of $y = u(x)$, and the Jacobian is the determinant of the kxk matrix of partial derivatives:

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_k} \\ \frac{\partial x_2}{\partial y_1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial x_k}{\partial y_1} & \cdots & \cdots & \frac{\partial x_k}{\partial y_k} \end{vmatrix}$$

PROOF. Denote by B the range of a transformation $y = g(x)$ with the inverse $x = h(y)$. Assume that $D \subset B$, and let C be the set of all points $x = (x_1, x_2, ..., x_k)$ that map into D under transformation. Therefore,

$$P[Y \in D] = \int \cdots \int_D f_Y(y_1, y_2, ..., y_k) dy_1 ... dy_k$$

$$P[Y \in D] = \int \cdots \int_C f_X(x_1, x_2, ..., x_k) dx_1 ... dx_k$$

$$P[Y \in D] = \int \cdots \int_D f_X(h_1(y_1, y_2, ..., y_k), ..., h_k(y_1, y_2, ..., y_k)) |J| dy_1 ... dy_k$$ □

If the transformation is not one-to-one, can be extended. The equation $y = g(x)$ has a unique solution over each set in a partition $A_1, A_2, ...$, and if these solutions have nonzero continuous Jacobians, then the pdf of Y is

$$f_Y(y_1, y_2, ..., y_k) = \sum_i f_X(x_{1i}, x_{2i}, ..., x_{ki}) |J_i|$$

EXAMPLE 3. Let X_1 and X_2 be independent and exponential, $X \sim EXP(1)$. Thus, the joint pdf is

$$f_{X_1,X_2}(x_1,x_2) = \exp\left(-\left(x_1+x_2\right)\right), (x_1,x_2) \in A$$

where $A = \{(x_1,x_2) \mid x_1 > 0, x_2 > 0\}$.

Consider the random variables $Y_1 = X_1$ and $Y_2 = X_1 + X_2$. This corresponds to the transformation $y_1 = x_1$ and $y_2 = x_1 + x_2$, which has a unique solution, $x_1 = y_1$ and $x_2 = y_2 - y_1$. The Jacobian is $|J| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$ and thus

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1, y_2 - y_1)\,|J| = \exp\left(-y_2\right), (y_1,y_2) \in B$$

and zero otherwise. The set of B is obtained by transforming the set A, an this corresponds to $y_1 = x_1 > 0$ and $y_2 - y_1 = x_2 > 0$. Thus $B = \{(y_1,y_2) \mid \infty > y_1 > y_2 > 0\}$.

The marginal pdf of $Y_1$and Y_2 are given as follows:

$f_{Y_1}\left(y_1\right) = ...$

$f_{Y_2}\left(y_2\right) = ...$

EXAMPLE 4. Suppose that, instead of the transformation of the previous example, a different transformation, $y_1 = x_1 - x_2$ and $y_2 = x_1 + x_2$ is considered.

The solution is $x_1 = ...$ and $x_2 = ...$ The Jacobian is $|J| = ...$ The joint pdf is given by

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(...,...)\,|...| = ... \in B$$

where $B = \{(y_1,y_2) \mid ...\}$. The marginal pdf of $Y_1$and Y_2 are given as follows:

$f_{Y_1}\left(y_1\right) = ...$

$f_{Y_2}\left(y_2\right) = ...$