

## Special Probability Distributions

The purpose of this chapter is to develop some special distributions. Special discrete distributions will be derived using the counting techniques. And Special continuous distributions also will be presented, and relationships between various special distributions will be discussed.

### 1.1. Special Discrete Distributions

**1.1.1. Bernoulli Distribution.** Suppose that a trial, or an experiment, whose outcome can be classified as either a success or a failure is performed. Let  $X = 1$  when the outcome is a success and  $X = 0$  when it is a failure, then the probability mass function of  $X$  is given by

$$(1.1.1) \quad \begin{aligned} p(1) &= P(X = 1) = p \\ p(0) &= P(X = 0) = 1 - p \end{aligned}$$

where  $p, 0 \leq p \leq 1$ , is the probability that the trial is a success. An alternative representation of this function is

$$f(x) = p^x q^{1-x}, x = 0, 1$$

and a random variable  $X$  is said to be a **Bernoulli random variable**.

In applications, Bernoulli random variables often occur as indicators. If  $A$  is an event, then the indicator random variable,  $I_A$  takes on the value 1 if  $A$  occurs and the value 0 if  $A$  does not occur

$$I_A(\omega) = \begin{cases} 1 & , \text{if } \omega \in A \\ 0 & , \text{otherwise} \end{cases}$$

Examine the properties of a Bernoulli random variable.

- $E(X) = \dots$
- $Var(X) = \dots$
- $M_x(t) = \dots$

**1.1.2. Binomial Distribution.** Let a more complicated experiment as a sequence of  $n$  independent Bernoulli trials. Each of which results in a success with probability  $p$  and in a failure with probability  $1 - p$ . If  $X$  represents the number of successes that occur in the  $n$  trials, then  $X$  is said to be a binomial random variable with parameters  $(n, p)$ . Thus, a Bernoulli random variable is just a binomial random variable with parameters  $(1, p)$ . The probability mass function of a binomial random variable having parameters  $(n, p)$  is given by

$$b(x; n, p) = \dots, x = \dots$$

EXAMPLE 1. It is known that screws produced by a certain company will be defective with probability .01, independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

Solution.

EXAMPLE 2. A communication system consists of  $n$  components, each of which will, independently, function with probability  $p$ . The total system will be able to operate effectively if at least one-half of its components function. (a) For what values of  $p$  is a 5-component system more likely to operate effectively than a 3-component system?

Examine the properties of a binomial random variable.

- $E(X) = \dots$
- $Var(X) = \dots$
- $M_x(t) = \dots$

**1.1.3. Geometric Distribution.** The geometric distribution is also constructed from independent Bernoulli trials, but from an infinite sequence. Suppose that independent trials, each having a probability  $p, 0 < p < 1$ , of being a success, are performed until a success occurs. Let  $X$  equal the number of trials required, then  $X$  is said to be a geometric random variable with parameters  $(p)$  with the probability mass function

$$P(X = x) = \dots, x = \dots$$

EXAMPLE 3. An urn contains  $N$  white and  $M$  black balls. Balls are randomly selected, one at a time, until a black one is obtained. Assume that each ball selected is replaced before the next one is drawn, what is the probability that (a) exactly  $n$  draws are needed? (b) at least  $k$  draws are needed?

Examine the properties of a geometric random variable.

- $E(X) = \dots$
- $Var(X) = \dots$
- $M_x(t) = \dots$

**1.1.4. Negative Binomial Distribution.** Suppose that independent trials, each having probability  $p, 0 < p < 1$ , of being a success are performed until a total of  $r$  successes is accumulated. Let  $X$  equal the number of trials required, then  $X$  is said to be a negative binomial random variable with parameters  $(r, p)$  with the probability mass function

$$P(X = x) = \dots, x = \dots$$

EXAMPLE 4. If independent trials, each resulting in a success with probability  $p$ , are performed what is the probability of  $r$  successes occurring before  $m$  failures?

Examine the properties of a negative binomial random variable.

- $E(X) = \dots$
- $Var(X) = \dots$
- $M_x(t) = \dots$

**1.1.5. Hypergeometric Distribution.** Suppose that a sample of size  $n$  is to be chosen randomly (without replacement) from an urn containing  $N$  balls, of which  $m$  are white and  $N - m$  are black. Let  $X$  denote the number of white balls selected, then  $X$  is said to be a hypergeometric random variable with parameters  $(n, M, N)$  with the probability mass function

$$P(X = x) = \dots, x = 1, 2, \dots$$

EXAMPLE 5. A purchaser of electrical components buys them in lots of size 10. It is his policy to inspect 3 components randomly from a lot and to accept the lot only if all 3 are nondefective. If 30 percent of the lots have 4 defective components and 70 percent have only 1, what proportion of lots does the purchaser reject?

Now, Examine the properties of a hypergeometric random variable.

- $E(X) = \dots$
- $Var(X) = \dots$

**1.1.6. Poisson Distribution.** A random variable  $X$  that takes on one of the values  $0, 1, 2, \dots$  is said to be a Poisson random variable with parameter  $\lambda$  if, for some  $\lambda > 0$ ,

$$P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}, x = 0, 1, 2, \dots$$

The Poisson random variable has a tremendous range of applications in diverse areas because it may be used as an approximation for a binomial random variable with parameters  $(n, p)$  when  $n$  is large and  $p$  is small enough so that  $np$  is of moderate size. The number of successes occurring is approximately a Poisson random variable with parameter  $\lambda = np$ . Examine the properties of a poisson random variable.

- $E(X) = \dots$
- $Var(X) = \dots$
- $M_x(t) = \dots$

**1.1.7. Discrete Uniform Distribution.** A random variable  $X$  that takes on one of the values  $1, 2, 3, \dots, N$  is said to be a discrete uniform random variable with parameter  $N$  if, for some  $N > 0$ ,

$$P(X = x) = \frac{1}{N}, x = 1, 2, \dots, N$$

Examine the properties of a discrete uniform random variable.

- $E(X) = \dots$
- $Var(X) = \dots$
- $M_x(t) = \dots$