Constructing Mathematical Concepts through External Representations Utilizing Technology: An Implementation in IRT Course

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Abstract **– One of useful theories in the educational and psychological assessment is an item response theory (IRT) that applies mathematical concepts. For students, constructing math concept to solve problems in IRT is very difficult and utilizing technology such as computer and software will help. In a qualitative approach, this study described how students constructed mathematical concepts in the item response theory using external mathematical representations integrated with technology. The participants were 19 graduate students. Data were collected through observation in 5 courses. The data were analyzed using Milles and Hubberman technique through reduction, display, and verification. The results showed that by using external representations and providing cases, presenting data in tables, the collaboration among the students to solve problems, create and demonstrate graphics utilizing technology to construct mathematical concepts, utilize the context assessment related to mathematical concepts, and also use graphs could support students' thinking for constructing mathematical concepts in IRT.**

Keywords **– External representation, constructing mathematical concepts, item response theory, technology integration.**

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1. Introduction

In psychometric lectures at postgraduate studies, one of the theories that must be studied is the item response theory. In this theory, the relationship between the latent ability acquired by respondents with the probability to answer correctly an item of instrument is explained using mathematical equations. The relationship between the latent trait and the probability of correct answers is also correlated to the item parameters, such as the item difficulty level, the item discriminating index, and the pseudo-guessing index. It is not a linear relationship but a nonlinear relationship. The mathematical equations used and these nonlinear relationships led to the theory of the item response including the difficult courses, especially when associated with the utilization and application of the item response theory in the field of psychometrics and education.

Various IRT applications in the field of psychometrics and education can be done. First, the participant's response to an instrument can be used to estimate the participant's ability and the item parameter. It is done after testing the assumptions that are unidimensional assumptions, local independence, and parameter invariance. Second, the item parameter can be used to identify the item bias or it is known as the differential item functioning. Third, if a test administration uses a battery test, the researcher is able to know the equality of the test devices (equating). Fourth, the researcher can develop the question bank and then uses it in the computer adaptive testing by using the parameter of the item. IRT can be applied either to unidimensional or multidimensional data, as well as to some data types, both dichotomous and polytomous. The more item parameters and ability parameters are used, the more complex the relationship between the parameters becomes.

The nonlinear relationship between the item parameters and the ability becomes a challenge in studying IRT. It is also the same as the process of the estimating item and ability parameters. It is applied to large data to solve psychometric and educational problems. Another challenge is that IRT intersects with applied mathematics as well as statistics, which requires some basic mathematical concepts to learn and use. Hence, mathematical representation is an indispensable skill for studying IRT.

Representation is the process of modeling concrete objects in the real world (objects) into abstract concepts or symbols and relationships among them [1]. [2] described the representation as a creativity that involves the expression and expression of ideas and feelings and the use of various ways to do so. It is supported by [3] that the ability must be acquired in order to allow students to use representations to organize, record, communicate mathematical ideas, apply and translate through mathematical representations to solve the problems and model as well as interpret mathematical phenomena. It shows that the representation is important in problem solving. In solving a mathematical problem, one needs to understand the problem, plan the problem solving, do the problem solving, and then interpret the solution into colloquial language. Understanding the problem to be solved and communicating the solution in colloquial language are related to the ability of mathematical representation. It is in accordance with the statement proposed by [4] that representational competence is the ability to use the representation to understand and communicate mathematical ideas to solve problems. The representation of students is to solve problems in mathematics [5].

There are two types of representations, namely internal and external representations [6]. An internal representation is related to the structure of knowledge within humans, which includes proportional, analogical, procedural, parallel, and distributed representations. An external representation is related to the structure and knowledge in the environment, including physical symbols, objects, images, etc. The role of the external representation is to present information, to be analyzed and processed during learning [6]. For problem solving, these two types of capabilities can be switched and interpreted between interpretations [4], based on the needs. Both of these representational competencies can be mastered by students through learning processes.

Before starting a lesson, a teacher/lecturer plans how to master the concept of mathematics and problem solving skills. The planning and execution of strategies for the formation of this knowledge structure are as a part of the internal representation. In planning the lesson, the lecturer should plan how the structure of knowledge, such as representing discrete symbols, performing analogical representations, understanding procedures and processes, and also representing at any discrete place in memory but instead is distributed over a large set of representing units. To facilitate student' representation, this learning can be supported by the use of external representations.

External representation used in learning can be in the form of diagrams, graphs, pictures which are used in cognitive tasks, both reasoning and problem solving as well as decision making [6]. This representation is very useful in learning. The use of external representations (diagrams and models) is used to build cognition and processes in biology [7]. External representation encourages students' active participation to build models in teaching [8]. External representation makes people think more powerfully than without using it [9].

Regarding the internal and external representations in learning, both are closely related. [10] stated that the learning process which is based on the internal representation developed by the students is an inference or a product of the external representation or vice versa. Meanwhile, [6] argues, "External representation can be transformed into internal representations by memorization." Based on these two opinions, external representations can be transformed into internal representations and are reversible. Utilizing both types of representations and their relationship become an assurance to students to participate actively in understanding the concept during learning, thinking more powerfully, solving problems, and making decisions. This reinforces the need for the use of external representation to address the problems experienced by the lecturer in learning.

Based on the experience of the researchers who teach IRT for 8 years, it is concluded that this course is challenging. The learning reflection already conducted shows that the students do not understand much about the mathematical concepts found in the lectures. In addition, the students also get difficulties in using these concepts to solve the problems in psychometrics. The students are heterogeneous coming from various majors (religion, language, science, and social science) and not all of them have ever studied statistics deeply. Constraints to conceptual understanding are caused by lack of clear representation of the concept [11]. The lecturer therefore needs to design a special learning to solve the existing problems by utilizing the external representation to form the internal representation and vice versa in order that the students are able to understand and apply the concept of IRT better.

Some previous studies indicate that the use of representation gives positive results. The utilization of the visual representation such as graphics, charts, pictures, etc., can improve students' success [12].

The similar result is also shown in another study by using the schematic representation positively associated with the success in problem solving [13]. The treatment group with the external representation used in studying the boxplot gets the higher score than the control group [14]. The use of the graphical representations helps college students in problem solving and the concepts are more difficult presented with the symbolic format than graphical format on algebraic problems [15]. The use of visual representation in learning improves student performance and reduces cognitive load [16]. The use of multi representational learning materials has a positive effect on learning [17]. Multiple representation-based learning improves students' ability to solve problems in algebra [18]. Students attempt to solve the probability problems using the external representations such as drawings, schemes, Venn diagrams, and contingency tables [19].

In the contemporary era, making graphics, images, etc., for the purposes of mathematical representation is not difficult anymore. For the purposes, users can use computers/laptops and software. In teaching and learning process, teacher/lecturer can integrate the utilization of technology to help students make mathematical representations. Educators can take advantage of this technology to facilitate students in the process of constructing knowledge and mastery of concepts through mathematical representation. This convenience will reduce the cognitive burden of students in learning. The technology integration can improve teaching and learning affectivity [20].

Various factors influence the successful use of the external representation in learning. The effect of utilizing the visual representation of the information related to the problem solving depends on the student ability, especially on low-achieving students and this visual representation is also strongly correlated to the problem solving compared to the average-achieving students [21], also the learning approach that is used in the classroom [22]. Based on the results of 25 studies [23], the mathematical representation is an evidence-based strategy for students with learning difficulties in problem solving.

2. Methods

The study described how students constructed mathematical concepts in the item response theory using external mathematical representations and integrating technology. This study utilized a qualitative approach. At the beginning of the study, the researchers designed a trajectory of learning developed by [24]. This design was equipped with tasks containing the external representations which were then developed into a lesson plan. Using this lesson plan, the researchers conducted an active learning integrating with technology. The process of forming the concept was observed and then recorded in the field notes for data analysis.

The data of the study were qualitative in phenomenological nature about how the students constructed mathematical concepts in the item response theory using the external mathematical representations. The data were collected through observations in 5 courses. The participants were 19 graduate students consisting of 9 males and 10 females. The students were from various majors (multi-entry), from science, language and also social science. The data were analyzed using the Miles and Hubberman model [25]. The stages of analysis were data reduction, data presentation, data verification and conclusion.

3. Result

The results of this study described the learning activities in IRT courses using the external representations and integrating with technology, such as computer/laptop and many software in item response theory. The activities were conducted in five different meetings with different materials including investigating the weaknesses of the classical test theory, understanding the IRT mathematical model, estimating the parameters, assuming IRT, and matching the model. The learning process in IRT with the external representation for each meeting is outlined as follows. All of teaching and learning processes had been implemented utilizing technology, especially many software installed in personal computer/laptop.

Investigating the weakness of the Classical Test Theory

Activities to investigate the weaknesses of classical test theory used a mathematical representation approach that was by performing simulations using the empirical data. The empirical data used for the simulation materials were the national exam results data on the mathematics subjects and science of Junior High School. In this simulation, students were divided into four groups. Two groups were asked to perform simulations using the mathematical data and two other groups were asked to perform simulations using the science data. From each data obtained, each group was asked to take samples from schools with the high achievement identities and schools with low achievement identities. The schools with the high achievement identities represented the high-performing group or examinees, whereas the schools with the low achievement identities represented the low-ability groups. The data from each group (high and low) were then analyzed using the item analysis software to determine the value of the item parameters (difficulty level [p] and discriminant [rpbis]) and estimated the instrument reliability (α).

Furthermore, the parameter values for the high and low groups were presented. The simulation results of each group are presented in Table 1.

Table 1. The simulation result using the Classical Test Theory.

Group/Analyzed	Sample Group	Parameter		
		α	p	rpbis
I/Science	High	0.90	0.73	0.75
	Low	0.91	0.47	0.48
II/Science	High	0.60	0.89	0.29
	Low	0.79	0.70	0.34
III/Math	High	0.94	0.86	0.58
	Low	0.93	0.64	0.53
IV/Math	High	0.91	0.78	0.49
	Low	0.86	0.58	0.40

The students were then asked to represent the results of the analysis. From the simulation result, it could be concluded that although using the same set of items/problems, the high and low groups produced different parameter values. For example, from Table 1. it could be seen that the difficulty level parameter was higher if the sample used had a higher ability than the average ability of the population. Then the discriminating item and reliability parameters also varied for each sample group, although the item/problem used was the same. From these findings, the students concluded that the statistical formulas used in the classical test theory such as the difficulty level, the discriminating item, and the test reliability were highly dependent on the samples used in the analysis. From the simulation and interpretation of data conducted, the students were able to investigate the main weakness of the classical test theory called the Dependent Group weakness.

Understanding the Mathematical Model of the Item Response Theory

The mathematical model in the item response theory consisted of one-parameter model (Rasch model), two-parameter model and three-parameter model. The parameters used in the one-parameter model involved only the difficulty level parameter (b). In addition, the two-parameter model used the difficulty level parameter (b) and discriminating index (a), while the three-dimensional model used the additional parameters besides those two parameters namely, pseudo guessing (c). To understand each model approach, it was representation of each characteristic curve of the items of each model. To create the item curve characteristic (ICC), the students were asked to perform simulations using MS Excel program by substituting parameter values into the mathematical model equations. For example, the most complex mathematical model of the item response theory was the 3-parameter equation model. In this case, the student was asked to do the simulation using MS Excel. In this simulation the students were asked to input each parameter values a, b, and c to the model equation 3-parameter, then next the students were asked to make ICC from model 3-parameter and the result was presented in Figure 1.

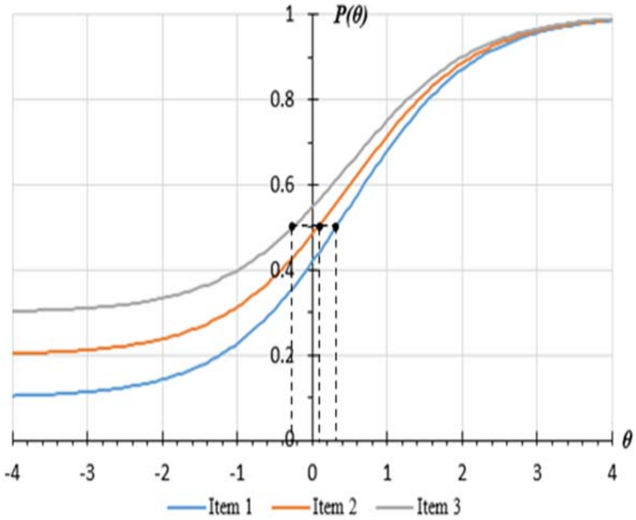

Figure 1. ICC Model 3 parameter with Item 1 (b = 0.5, a = 1, c = 0.1), Item 2 (b = 0.5, a = 1, c = 0.2), and Item 3 (b = 0.5, a = 1, c = 0.3).

The students were then asked to identify the characteristics of the model 3 parameter equations based on the ICC in Figure 1. From Figure 1., the students understood that the parameter value of c would affect the ability required to correctly answer a question item. Then the students were informed that with the same values of a and b, the greater the parameter c became, the higher the chance they had to correctly answer an item. For example, to answer correctly with a probability of 0.5 (50%) using an item with the same difficulty level and dissipation, it requires the ability on a scale of 0.32 if the parameter value of c = 0.1, on a scale of 0.1 if the parameter value c = 0.2, and on a scale of -0.3 if the parameter value c = 0.3 (Figure 1.). This showed that the students were able to understand the relationship between the parameters of the ability and the item parameter contained in the mathematical model of 3 parameters. To understand the relationships among parameters, on model 1 and 2, activities needed to be performed are exactly the same, by simulating using the real data and interpreting each curve of the characteristic test obtained.

Understanding the Parameter Estimation in the Item Response Theory

In the item response theory, the item and ability parameters could be estimated manually or by using software. To understand the process of the parameter estimation, the learning activity was simulated using the student response data and with the help of MS

Excel. The following table presents an example for the four items of the multiple choice questions obtained from the student responses.

Table 2. The example of the student response data.

Respondent	Item			
	1	2	3	4
S_1	1	0	0	1
S_2	1	1	1	0
S_3	1	0	1	1
S_4	1	1	0	0
S_5	0	0	1	0
S_6	0	1	0	0
S_7	1	1	1	1
S_8	0	0	0	0

Based on the table, it could be seen that the pattern of the student answers was different. The simulation was then performed for the model of the equation 1-parameter, where the parameter value of item b was determined first, for example for the four items, the parameter value b was 0.5; 0.5; 0.6; and 1. Further calculations were made using the likelihood function using MS Excel. The same procedure was also done to estimate the item parameters in the equation model 2-parameter and 3-parameter. The following was presented as one of the results of the simulation activity of the ability parameter estimation of the model equation 1-parameter.

Table 3. The result of the simulation of the Ability Parameter Estimation (Rasch Model).

Respon-dent	Answers	Correct Answers	Values MML*	Ability (θ)
S_1	1001	2	0.05009	0.65
S_2	1110	3	0.14723	1.75
S_3	1011	3	0.08929	1.75
S_4	1100	2	0.08259	0.65
S_5	0010	1	0.10918	-0.45
S_6	0100	1	0.12067	-0.45
S_7	1111	4	0.86856	2.25
S_8	0000	0	0.96193	-0.95

* Marginal Maximum Likelihood

Next, the students were asked to represent the data from the simulation result. Considering the simulation process and the result, the students could conclude that for model 1-parameter, the distribution of the test takers' answers did not affect the ability scale. For example, students S_1 and S_4 had a different correct answer distribution, but the number of their correct answers was the same, 2 points. The ability scale for both test takers was also the same, 0.65. Another thing revealed through the simulation was that the parameter value b for model 1-parameter also had no effect on the parameter value of the test takers' ability. For example, in this simulation, the four items used had the values of the different parameter b, but the ability parameter values were the same for the test takers with the same number of correct answers. The same activity was also done to understand the process of the parameter estimation on the logistic model 2-parameter and 3-parameter. From the simulation activities conducted and the conclusions made by the students, it appeared that the students were able to understand the estimation of the ability parameters in the item response theory using the likelihood function.

Understanding the Assumption of the Item Response Theory

In the item response theory, there were three assumptions that had to be fulfilled, namely unidimension, local independence, and parameter invariance. To understand the assumptions applying in the item response theory, the student used the real data and then presented the results of the analysis with pictures. To understand the unidimensional assumption, the students were asked to do the factor analysis using SPSS software. This factor analysis aimed to look at the eigenvalues in the inter-item variant-covariance matrix and began with an analysis of the adequacy of the sample. Information on the adequacy of the sample was a prerequisite for further analysis. After obtaining the eigenvalue of the analyzed data, the students were then asked to present it with a scree plot. One of the scree plots produced by one of the students is presented in Figure 2.

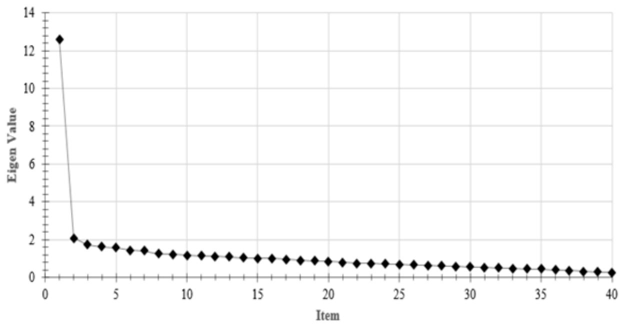

Figure 2. Scree-plot eigenvalues

The students were then asked to represent the scree plot. They were asked to observe the number of the steep slopes on the scree plot. From the scree plot presented in Figure 2., it could be seen that there was only one peak. Meanwhile, starting from the component 2 until the next components, the graph continued to slope down. From the scree plot, the student concluded that there was 1 dominant factor in the test device being analyzed. From these findings, the students came to a conclusion that the analyzed test device only measured 1 dimension. The result of the scree plot was one example that the unidimensional assumption was fulfilled. From this activity, it appeared that the students were able to understand and prove the unidimensional assumption on the theory of the item response through the presentation of the eigenvalue on the scree plot and also to represent it.

The external representation approach was also used to understand the assumptions of the item and ability parameters invariance. To understand and prove this assumption, students were asked to find the real data to be analyzed. After obtaining the data, the students were asked to divide the data into two parts, such as the data grouped by sex (male and female groups) or the data grouped by splitting the test device into two parts (for example there were 40 items: item 1 - 20 as the first half, while item 21 - 40 as the second half). The students were then asked to estimate the item and abilities parameters of the data groups using software (Bilog, Quest). After obtaining the value for each parameter, the student was asked to present it in the form of the scatter diagram. To see if the distribution of points on the scatter diagram has a strong correlation, the student was required to draw a straight line with a gradient of 1 on the scatter diagram resulted. The following figure is one of the scatter diagrams done by the students for parameter b.

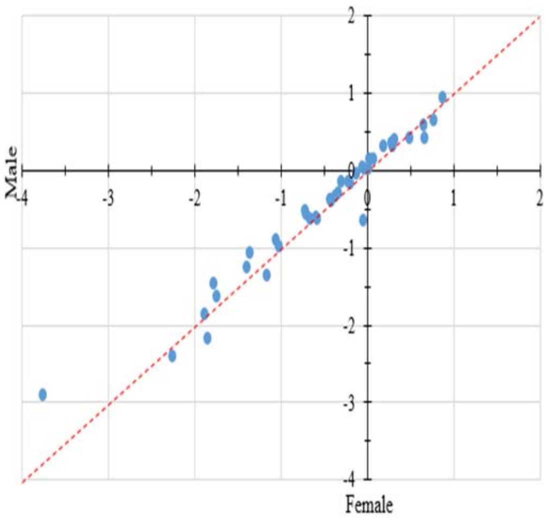

Figure 3. The Scatter diagram for parameter b

To understand the concept of the parameter invariance, the students were asked to represent the scatter diagram. By looking at the result of the scatter diagram, the student concluded that the assumption of parameterized invariance b was fulfilled since the distribution of dots was in a straight line with a gradient of 1. This indicated that the student was able to understand the concept of the parameter invariance assumption. The same process was done to prove the invariance of parameter a and c.

Next, to understand and prove the assumption of the ability parameter invariance, the students were required to estimate the ability parameters of the previous data groups. For example, in this activity the student estimated the ability parameters for the two-part test, the first half items (1 to 20) and the second half items (21-40), and then presented their estimation results into the scatter diagram. To see the correlation of the distribution of the dots in the plot, a straight line with a gradient of 1 was made on the scatter diagram. The following is one of the scatter diagrams of the ability parameter.

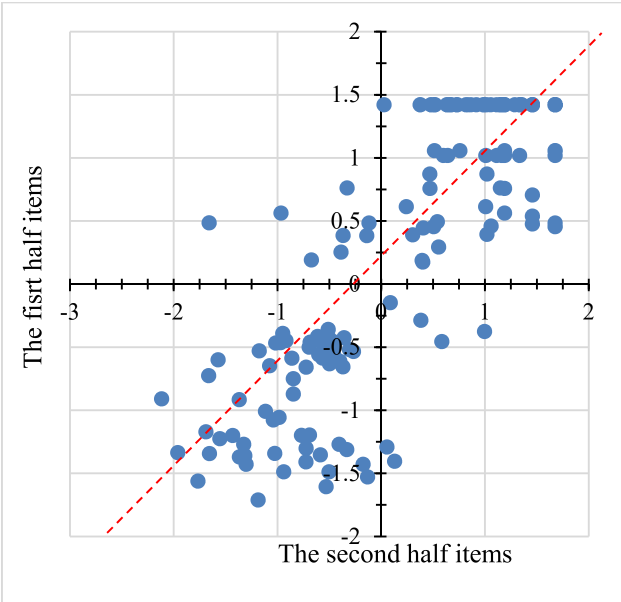

Figure 4. The scatter diagram of the ability parameters

After that, students were required to observe and represent the scatter diagram. The students then concluded whether the assumption of ability parameter invariance was fulfilled or not. The keyword in verifying this assumption was that if the distribution of dots on the scatter diagram tended to approach the straight line with a gradient of 1, the assumption of ability parameter invariance was fulfilled. Thus, by starting with the parameter estimation, then making the scatter diagram and representing it, the students were able to understand and prove the assumption of item parameter invariance and ability.

Determining the Goodness of Fit with the Item Response Theory Model

To understand the concept of the goodness of fit with model in the item response theory, the external representations using the real data could be done. There were two ways to determine the most suitable model, first, by determining the value of chi-square and second, by making the characteristic curve plot. Both ways could be done using various software. For example, to determine the most suitable model (1 PL, 2 PL, or 3 PL), the students were guided to analyze the data they had using of BILOG software and to represent the results of the analysis. Table 4. below presents one of the analysis results using the model 1-parameter, 2-parameter, and 3-parameter for one of items (of IPA01 data).

Table 4. Comparison of $\chi 2$ values logistic model 1 PL, 2 PL, and 3 PL.

Value of $\chi 2$						
1-parameter			2-parameters		3-parameters	
ITEM	CHISQ (PROB)	DF	CHISQ (PROB)	DF	CHISQ (PROB)	DF
IPA01	46.8 (0.0000)	8.0	28.6 (0.0004)	8.0	5.8 (0.6671)	8.0

After obtaining the results, the students were asked to look at the criteria of the model appropriateness, i.e. the value of $\chi 2$ table $<\chi 2$ count or $\alpha > 0.05$. From the results of the analysis and the existing identified criteria, the students concluded that the most suitable model for item 1 was the model of 3-parameter.

The second way to see the model suitability was by identifying the plot of the item characteristic curve. In this case, the students were asked to display the item characteristic curve through BILOG software. They were asked to see how well the data distribution fit the model. The following Figure is an example of the IPA01 item characteristics curve using the model of 1-parameter.

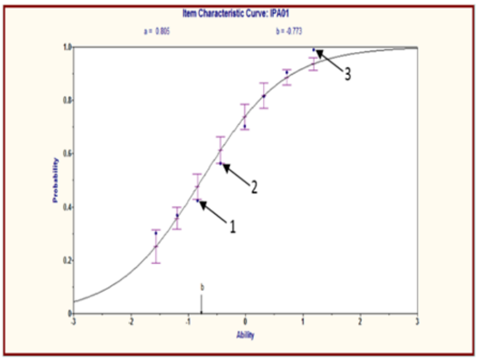

Figure 5. IPA01 Item Characteristic Curve using the Model of 1-Parameter.

Next, the students were asked to observe the data distribution on the curve. Figure 5. showed that there were some dots whose position was far from the model curve (dots 1, 2, and 1). Based on the graphic, it meant that the model 1-parameter was not suitable for IPA01-items' analysis.

To analyze the distribution of the suitable data model, the students were asked to display the item characteristic curve: IPA01 which was analyzed with the model 3-parameters, as presented in Figure 6.

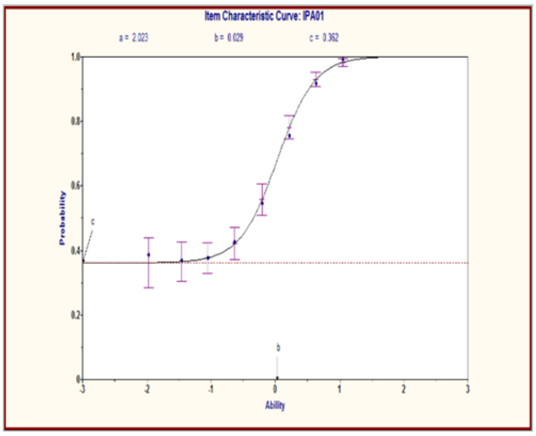

Figure 6. The Item Characteristics Curve: IPA01 with the model 3-Parameters.

Figure 6. showed that the data distribution approached the model 3-parameters, where the dots came close to the curve. This indicated that the model 3-parameters were suitable for the items analysis of the IPA01 data set. By comparing the characteristic curve among the models of 1-parameter, 2-parameters, and 3-parameters, the students could determine the most suitable model for the data analysis. Based on the activity done, the students could understand the suitability of the model by interpreting the data analysis result (comparing the chi-square value with the criteria), or representing the item characteristic curve (see the data distribution).

To get a direct response from the students regarding the use of external representation in understanding the mathematical concepts in IRT, the researcher asked the students to give their opinion. Here are the opinions of some students about IRT lectures that had been held.

> "... using an external representation, I found it easier to understand the formulas contained in the IRT. Previously I found it difficult to understand the formulas, because my previous educational background was not from mathematics, but using the external representations, it helped me to understand the formulas and relationships among parameters contained in the formula ... "(Student 1)
>
> "... it helped me to understand the IRT concepts using the external representations, especially to understand mathematical equations ... using visualizations in graphs and images, I became aware of the relationships among parameters ... thus I also understood the direct use of a concept learned ... "(Student 9)
>
> "... the use of external representations helped me much to understand the IRT applications... this kind of a learning model made my friends and me to do a direct simulation of the IRT concept

implementation, this gave a direct experience for us, just because we did not get used to with the data management on a large scale, I found it rather difficult if the lecturer did not guide us ... "(Student 11)

"... the presentation of tables, graphs, and drawings helped me much to understand the IRT concepts ... after a mathematical equation visualized in the graphic form, I could just understand the goal and use of the equations. I understood, applied, and analyzed it easily... "(Student 18)

These opinions indicated that the students responded positively to the learning implementation with the external representation. In general, students found it easier to understand the IRT concept when the concept was represented in other forms such as with the presentation of tables, drawings, curves, and so on. The direct involvement through the simulation activities creates the learning activities more meaningful and gave the direct experience for students in applying the concepts they had learned. The challenges often faced by the students were that they did not get used to do the data management on a large scale, so that the teacher still needed to guide them technically. Another challenge, several non-mathematics' and science' students got difficulty to relate some mathematical concepts in high school in the item response theory, for example the concept of equations, derivatives and integral similarities, which were solved by discussing the concept specifically. However, in general, the learning activities run smoothly.

4. Discussion

Various challenges are indeed encountered in IRT learning. This is because IRT directly intersects with applied mathematical concepts and statistics, requiring some basic mathematical concepts to learn and use. For example, in the logistic equation model 1 PL, 2 PL, and 3 PL, the models are presented in the form of a mathematical model that contains formulas relating to the relationship among parameters. For people whose educational background was not mathematics or statistics, it was difficult to understand the relationships between parameters formulated in the mathematical model. Associated with this, representing the mathematical model in other forms was indispensable to facilitate in the study of IRT.

Based on the research findings, the use of real data as a simulation material was helpful to understand the concept of IRT. With the availability of this data students could directly practice to conduct analysis related to IRT concepts. In addition, the practice also made students more motivated than when they were asked to understand a concept or formula by reading a book. The need for mathematical representation began when students started managing the existing data, because the data provided was still raw data, so it still needed to be processed. Furthermore, with the direction and guidance of lecturers, students performed the data analysis in accordance with the concept being studied. Then the results of the analysis were presented either in the form of tables, graphs or drawings. The results obtained in the form of tables, graphs, and images were then represented using the language itself by students with the direction of the lecturer. Through the presentation of tables, graphs, or images was easier for students to understand a concept in the IRT. Utilizing technology, students created external representation easily. For example, the student was able to explain the relationship between the item parameters (b, a, and c) and the ability (θ) in logistic model 1 PL, 2 PL, and 3 PL by looking at the item characteristics curve. Students were also able to explain the influence of one parameter against another parameter, if the value of one parameter was maximized or minimized. This showed that by using the external representation through the graphical representation, it made it easier for students to understand the mathematical model of IRT logistic equations. This was in line with the opinion put forward by [6]; [7], that external representation in learning could help build cognition and process in order to easy understanding the concept.

There were several reasons why students were easier to understand the relationship equation between parameters in the IRT logistic equation. First, the use of external representations of visualization in the form of images, where the formulas that existed in the logistic equation model transformed in the form of the item characteristics curve, could reduce the burden of student thinking, as had been proven through the research results [16]. Secondly, by simulating, for example by maximizing or minimizing the value of each parameter, students could observe directly the changes that occurred on the item characteristics curve. This made it easier for students to see the relationships between parameters and made it easier for students to interpret each characteristic item curve they had made. Third, the students' 'reluctance to interpret the item characteristics curve reduced the students' thinking burden in understanding the IRT logistic equation model.

The presentation of data through the table also helped students to understand the concept of IRT. This was in line with the opinion of [19]. In learning to investigate the weaknesses of the classical test theory, the results of analysis from several groups were presented in the tabular form. The table made it easier for students to compare the analysis results of each group from different data groups. By looking at the data in the table and making comparisons based on certain criteria, students were able to explain the weaknesses of classical test theory and the things that caused it. Students were also able to mention the weaknesses of the classic test theory with their own language. This suggested that there was a deep understanding of the concepts being studied. Thus, the presentation of data through the table as another representation of the analysis results made it easier for students to understand the concept. In addition, simulated activities guided by lecturers were also very helpful for students to understand the use of a formula. For example, in estimating the ability parameters using the Likelihood function, the function was very difficult to understand, but by doing a simulation with the help of MS Excel the students were able to explain the relationship and influence of each parameter contained in the function. This finding was certainly in line with the results of the researches [12] and [13], that the use of visual and schematic representations could improve students' success in understanding the concept.

The use of external representations in IRT learning was able to facilitate students in learning by doing, both for problem solving activities (case studies) and simulations. The concepts studied were not only partially understood, but they were comprehended comprehensively. This comprehensive understanding referred to how a concept was shaped up to its application. A representation approach through the use of real data and a context assessment allowed it to occur. It also provided evidence that the principle of realistic learning was embraced in the process of the conceptual understanding. Thus, the ability of the external representation was related to the principle of the realistic learning. This was one of them that could lead students to understanding the concepts more meaningfully and comprehensively. Various study results supported these findings including the results of the research [17].

Another aspect of concern in the use of the external representation in IRT learning was related to the process of communicating the concept. The ability of the external representation of students greatly affected the process of communicating the concepts that had been studied. When students presented in front of the class, they were able by communicating to explain a concept using their own language, without having to fixate on the definitions and mathematical formulas contained in textbooks. The presentation activity was usually followed by the discussion (questions and answers), with the ability of representation that had been trained during the learning process, communication between students became more flexible, because they conveyed a concept, ideas using things they thought was easiest to understand, such as through the presentation of tables, graphs, etc. This finding was in line with the opinion of [3], that representation was needed to improve communication skills.

If it referred to the learning process that had been done, the tracing external representation in IRT learning could spur the students to construct their own understanding. Previous studies had shown that using the external representations would have a positive impact and trigger students to solve the problems given [18]; [19]. In this case, the role of the teacher was not dominant, the teacher was only responsible for managing the activities to be performed and do the division of tasks. The dominant role of students in constructing their knowledge made the learning process more meaningful and the knowledge gained could be mastered and understood in depth. This suggested that when understanding a concept, it should not be too fixed by conventional means (e.g. memorizing formulas or definitions), but representing it in other forms (presented in tabular, graphic or chart form); it helped students to understand the concept. This could certainly reduce the burden of student thinking and other advantages of applying the concept learned became easier, as previously stated by [16].

The constructive aspects described earlier provide an understanding that the principle of constructivism learning strongly influenced students' understanding of the concepts studied in the IRT. Many strategies could be chosen by lecturers to apply the principles of constructivism, such as problem-based learning, case studies, cooperative learning, and so forth. In this case, the use of the external representation could be one of the strategies in implementing constructivism learning. Besides, to improve student representation skills, it could also improve students' elaboration and communication skills. Another important point was that the use of the external representations was also capable of training students' mathematical connection skills, so mathematical concepts studied in IRT could be applied in many other disciplines, which focused only on theories of measurement and assessment.

5. Conclusion

The results of this study show that collaboration among the students to solve problems, create and demonstrate graphics to construct mathematical concepts, utilize the context assessment related to mathematical concepts, and using graphs could support students' thinking for constructing mathematical concepts in IRT. For educators who teach applied mathematics, it is recommended that the utilization of trajectory in learning and the external representation can facilitate students in obtaining knowledge used in problem solving. Concerning that some students with non-mathematics and science educational backgrounds have difficulty in connecting mathematics concepts in high schools to item response theory, for example the concept of line, derivative and integral equations, a special study is needed to deal with this problem. Future studies need to address identification of students' difficulties in mathematical connections in detail and also teacher strategies to overcome the problems, particularly in IRT lectures.

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