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The role of PWWs in helping students to prove the cosine of a sum

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Abstract. In learning mathematics, students learn to compile a mathematical argument and proof. However, the objects studied in mathematics are abstract objects. Sometimes, the visualization of the object is required to make students easier in understanding it so that they can prove it. Proofs without Words (PWWs) is an image media that has the opportunity to play a major role in learning mathematics from the basic education level to higher level. This study aims to explore how third-year students majoring in mathematics education in prove the cosine of the sum with the help of PWWs. This study uses a qualitative approach. Forty five third-year college students majoring in mathematics education were shown the PWWs images. Then, they were given the related questions of the drawing to assist the students in understanding the formula of cosine of the sum. The analysis on the students' answer showed that PWWs provide visualization of the abstract mathematical concepts. However, in the process they need to be reminded about polar coordinate concept and the concept of distance between two points on the Cartesian coordinates to answer the questions. PWWs can be used to assist them in constructing an understanding and proof of the cosine of the sum. Nevertheless, it depends on the students' prior knowledge which is the basis for the concept.

1. Introduction

One of the competencies in learning mathematics is being able to compile arguments and prove a mathematical statement. In compiling a proof or argument regarding the truth of a mathematical statement, it is necessary to have ability to associate the characteristics and elements involved in mathematical statement. This mathematical ability is called adaptive reasoning [1]. A teacher has important task to help students to be able and understand how the argumentation and mathematical evidence are compiled [2,3]. Teaching reasons for mathematical argumentation and proof of mathematics taught to students are to: (1) students to get certainty from a mathematical statement; (2) to students to gain an understanding of mathematical concepts; (3) train students to communicate their ideas; (4) provide challenges for students to think deeply; (5) formulate a more general and systematic mathematical theoretical framework [4]. This statement was in line with the idea that mathematical learning should include three components. The three components were involving students in non-routine problem solving activities, facilitating students to analyse and evaluate and encouraging students to construct their own understanding [5].



One of teacher's tasks in learning mathematics is to help students to be able to compile a mathematical proof. The teacher or prospective teacher must also be able to prove mathematical theorems. Mathematical evidence can be very useful tools in the process of learning mathematics. There are seven functions of mathematical evidence in various learning activities. The functions are verification, explanation, discovery, communication, systematization, skills and theoretical predictions from empirical observations [6]. In reality, some teachers or prospective teachers only remember the form of mathematical theorem but they have difficulty in proving it, for example the formula of cosine of the sum $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$. Mathematics learning on trigonometric topics is closely related to geometrical material. In studying mathematical material related to the geometry, students often experience some obstacles that make them difficult to learn. These obstacles include poor understanding of geometric concepts, weak ability to identify geometry problems, weak ability to maximize information in solving problems related to students' geometry and weaknesses in utilizing visualization and representation in solving problems in geometry [7].

To prove this mathematical theorem, students must understand what is meant by the theorem. Theorem or the nature of mathematics is a mathematical object that is studied and is abstract. Consequently, students experience difficulties in understanding it. One way to overcome this problem is to make abstract mathematical concepts become more concrete by using image media to enable students think visually [8, 11]. One image media that can help students understand and prove a mathematical theorem is proofs without words (PWWs). PWWs is an image or diagram that only contains pictures/illustrations and mathematical symbols without containing a written sentence or statement [12, 13]. By observing PWWs students are expected to understand why a statement or theorem is correct then determine the first step in proving the theorem. PWWs are believed to have the opportunity to play a role in mathematics learning at the primary, secondary and higher education level [14]. When teachers make mathematics learning activities by providing various kinds of mathematical evidence about certain concepts, students have the opportunity to deeply understand the concept. When students communicate their understanding about mathematical evidence, it will create an atmosphere of discussion about the mathematical arguments. Hence, students can construct deeper understanding of the basic mathematical concepts that underlie that evidence [15]. However, the effectiveness in learning mathematical concepts using images depends very much on the students' prior knowledge about the concepts that have been learned before and how they can relate these concepts to the images they observe [8], [11]. If the students' prior knowledge and initial skills are weak, students will have difficulties in understanding the message contained in the pictures or visualization. On the contrary, if the students' prior knowledge and initial skills are strong, students will easily understand the message contained in the pictures or visualization so that they can construct their own understanding of mathematical concepts.

This study aims to explore how third-year students majoring in mathematics education prove the cosine formula of the number of angles with the help of PWWs.

2. Methods

This study uses a descriptive qualitative approach [16]. Forty five third-year students majoring in mathematics education were shown the PWWs images. Then, they were given the related questions of the drawing to assist the students in understanding the formula of cosine of the sum. Their works and answers were analyzed using descriptive qualitative approach. Forty five students were grouped into 9 groups consisted of 5 students. At the beginning of the activity, each student was given the following questions: Can you prove $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$? Then they were given PWWs as follows:

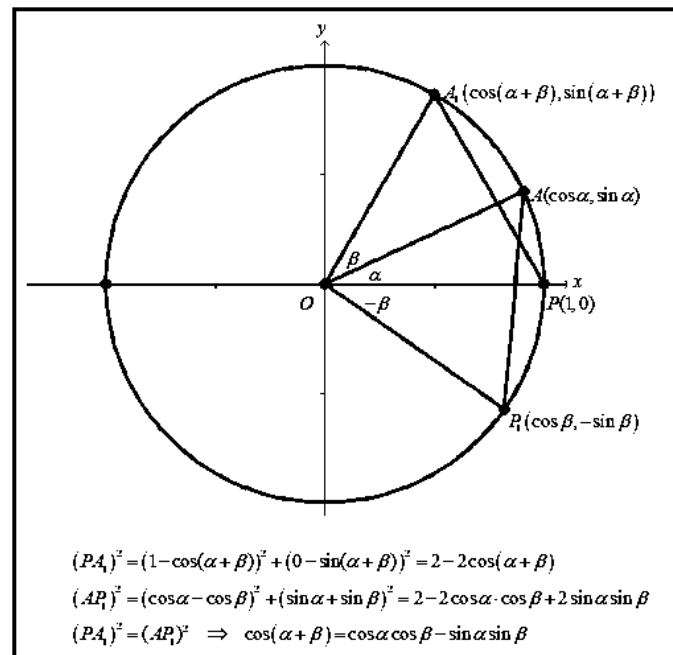


Figure 1. PWVs Pictures for Cosine of the Sum

To help them understand the picture and prove that $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$, they were given the following questions:

- (1) Is it true that the coordinate of A is $(\cos \alpha, \sin \alpha)$? Why?
- (2) Is it true that the coordinate of A_1 is $(\cos(\alpha + \beta), \sin(\alpha + \beta))$? Why?
- (3) Is it true that the coordinate of P_1 is $(\cos \beta, -\sin \beta)$? Why?
- (4) Is it true that $(PA_1)^2 = (1 - \cos(\alpha + \beta))^2 + (0 - \sin(\alpha + \beta))^2 = 2 - 2\cos(\alpha + \beta)$? Why?
- (5) Is it true that $(AP_1)^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 2 - 2\cos \alpha \cdot \cos \beta + 2\sin \alpha \sin \beta$ Why?
- (6) Is it true that $(PA_1)^2 = (AP_1)^2$? Why?
- (7) Is it true that because $(PA_1)^2 = (AP_1)^2$ then $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$? Why?
- (8) Based on your answer in number (1) to (7), prove that $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$.

3. Result and Discussion

The results of this study were that in the beginning all respondents could not prove that $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$. This showed that all respondents did not have a good understanding of the concept so that there were a chance of difficulties in helping students learn the concept [7]. Nine respondents stated that they knew the formula from the book, 21 respondents stated that they knew the formula from the teacher / lecturer explanation, while 15 other respondents said that they forgot where they got this information.

In question (1), the answer expected from the respondent was “because the length of the radius is 1 unit of length and $\angle POA = \alpha$. Hence, based on the comparison of true trigonometry that the coordinates is $(\cos \alpha, \sin \alpha)$ ”. In reality, almost all respondents experienced difficulties in answering these

questions, because they did not think that this question was related to the concept of polar coordinates. This reinforces the opinion that respondents' weakness in identification makes obstacles in solving problems [7]. This supports the theory that one's success in learning mathematics with visualization media depends on the person's initial knowledge and initial skills [8, 11]. After the respondent were reminded of polar coordinate concept, they began to fluently answer the question. 41 of the 45 respondents answered correctly and 4 respondents still gave wrong answers. Two of the four respondents, who had wrong answers, stated that the coordinates of A was $(\cos \alpha, \sin \alpha)$ because the radius of the circle was one unit of the length, while the other two respondents stated that the Pythagoras theorem can be found that the coordinates were $(\cos \alpha, \sin \alpha)$.

In question (2), the expected answer from the respondent is "because the length of the radius is 1 unit of length and $\angle POA_1 = \alpha + \beta$ hence based on the comparison of true trigonometry that the coordinates of A_1 are $(\cos(\alpha + \beta), \sin(\alpha + \beta))$ ". The reality that occurred was among 45 respondents, 43 respondents answered correctly and two respondents gave wrong answers. Both respondents stated that because the coordinates of A_1 must be from the x-axis, then the two angles must be summed up so that the coordinates of A_1 was $(\cos(\alpha + \beta), \sin(\alpha + \beta))$. They have weak understanding about concept of polar coordinates so that they had difficulty answering the question correctly. This was consistent with the opinion that weak understanding about concepts can hinder mathematical problems [7].

In question (3), the expected answer from the respondent was "because the length of the radius is 1 unit of length and $\angle POP_1 = -\beta$ because $\cos(-\beta) = \cos \beta$ and $\sin(-\beta) = -\sin \beta$ hence based on the comparison of the true trigonometry that the coordinates of P_1 is $(\cos \beta, -\sin \beta)$ ". In this study, 43 of 45 respondents, answered correctly and two respondents still gave wrong answers. One of the two respondents who answered wrongly stated that because the positive radius value of the positive x value, the coordinates of P_1 was $(\cos \beta, -\sin \beta)$. The other respondent stated that because the radius might not be positive then the coordinates of P_1 was $(\cos \beta, -\sin \beta)$. These two respondents have weak understanding on the concept of polar coordinates. Therefore, they have difficulty to answer this question correctly. This is consistent with the opinion that weak understanding about concepts can hinder mathematical problems [7].

In question (4), the expected answer from the respondent was "because the coordinates of A_1 is $(\cos(\alpha + \beta), \sin(\alpha + \beta))$ and the coordinates of P is (1, 0) then:

$$(PA_1)^2 = (1 - \cos(\alpha + \beta))^2 + (0 - \sin(\alpha + \beta))^2 \Leftrightarrow$$

$$(PA_1)^2 = 1 - 2\cos(\alpha + \beta) + \cos^2(\alpha + \beta) + \sin^2(\alpha + \beta) \Leftrightarrow$$

$$(PA_1)^2 = 1 - 2\cos(\alpha + \beta) + 1 = 2 - 2\cos(\alpha + \beta)$$

Thus, it is true that $(PA_1)^2 = 2 - 2\cos(\alpha + \beta)$ ". All respondents were able to provide answers as expected.

In question (5), the expected answer from the respondent was "because the coordinates of A is $(\cos \alpha, \sin \alpha)$ and the coordinates of P_1 is $(\cos \beta, -\sin \beta)$ then:

$$(AP_1)^2 = (\cos \alpha - \cos \beta)^2 + (-\sin \alpha - \sin \beta)^2 \Leftrightarrow$$

$$(AP_1)^2 = \cos^2 \alpha - 2 \cos \alpha \cos \beta + \cos^2 \beta + \sin^2 \alpha + 2 \sin \alpha \sin \beta + \sin^2 \beta \Leftrightarrow$$

$$(AP_1)^2 = 2 - 2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta$$

Therefore, it is true that $(AP_1)^2 = 2 - 2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta$ ". 44 of the 45 respondents in this study were able to answer correctly and one respondent did not answered.

In question (6) the expected answer from the respondent is that "because OA_1, OP, OA, OP_1 are the radius of the circle and the triangle A_1OP is congruent with the triangle A_1OP , so $(PA_1)^2 = (AP_1)^2$ ". 44 of the 45 respondents, answered correctly and one respondent did not give the correct answer, as he stated that because the coordinates of A_1 was $(\cos(\alpha + \beta), \sin(\alpha + \beta))$ then $(PA_1)^2 = (AP_1)^2$.

In question (7), the answer expected from the respondent was "because $(PA_1)^2 = (AP_1)^2$ then $2 - 2 \cos(\alpha + \beta) = 2 - 2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta$ so that $-2 \cos(\alpha + \beta) = -2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta$ so that $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ ". 42 of the 45 respondents, answered correctly and three respondents did not answered.

Questions (1) to questions (7) were questions that were expected to guide respondents to prove $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$. By being able to answer the question (7) correctly the respondent had actually proven the truth of the formula $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$. In question (8), respondents were asked to rewrite the evidence of $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ based on the results of their work in answering questions (1) to (7). However, among 42 respondents who answered question (7) correctly, only 17 respondents answered. 7 of them answered correctly, while the other 10 respondents were incomplete in writing their evidence. 25 of the 42 respondents did not answered the question. This shows that the ability of the majority of respondents in compiling mathematical arguments and compiling mathematical evidence in other words the ability of adaptive reasoning [1] of the respondents were not strong enough and need to be improved. If this ability was weak, it would be difficult for them to solve mathematical problems [7].

4. Conclusion

Based on the results of the analysis of the students' answer, it can be concluded that PWWs provide visualization of the abstract mathematical concepts of cosine of the sum, but in the process they need to be reminded about polar coordinate concept to answer the questions. PWWs can be used to assist them in understanding about proof of the cosine of the sum, but it depends on the students' adaptive reasoning proficiency to construct the formal proof of cosine of the sum, in which most of respondents here struggled in constructing the formal proof of cosine of the sum.

References

- [1] National Research Council 2010 *Adding + it UP! Helping Children Learn Mathematics* (Washington DC: National Academy Press)
- [2] Hanna G and de Villier M 2008 ICMI Study 19: Proof and proving in mathematics education *ZDM Math. Educ* **40** 329
- [3] Knipping C 2014 Challenges in teaching mathematical reasoning and proof – Introduction *ZDM Math. Educ* **14** 127
- [4] Hanna G 2014 *Mathematical proof, argumentation, and reasoning encyclopedia of mathematics education* (London: Springer)

- [5] Apino E and Retnawati H 2017 Developing instructional design to improve mathematical higher order thinking skills of students *J Phys Conf Ser* **812** 1-7
- [6] Herbst P G Miyakawa T and Chazan D 2010 Revisiting the functions of proof in mathematics classrooms: A view from a theory of instructional exchanges (Michigan: Deep Blue Univ Michigan) p 1
- [7] Retnawati H Arlinwibowo J and Sulistyarningsih E 2017 The students' difficulties in completing geometry items of national examination *Int. J. New Trends Educ. Their Implic* **8** 28
- [8] Giardino V 2010 Intuition and visualization in mathematical problem solving *Topoi* **29** 29
- [9] Alsina C and Nelsen R B 2006 *Math made visual* (Washington DC: The Mathematical Association of America)
- [10] Aso K 2001 Visual images as educational materials in mathematics *Community College J. of Res. Pract.* **25** 355–360
- [11] Brating K and Pejlar J 2008 Visualizations in mathematics *Erkenntnis* **68** 345-58
- [12] Nelsen R B 1993 *Proof without words: exercise in visual thinking* (Washington DC: The Mathematical Association of America)
- [13] Nelsen R B 2000 *Proofs without words ii : more exercises in visual thinking* (Washington DC: The Mathematical Association of America)
- [14] Alsina C and Nelsen R B 2010 An invitation to proofs without words *Eur. J. Pure Appl. Math* **3** 118-27
- [15] Knuth E J 2002 Proof as a tool for learning mathematics *Math. Teacher* **95** 486
- [16] Creswell J W 2012 *Educational research: Planning, conducting, and evaluating quantitative and qualitative research* 4th ed (Amsterdam: Pearson Education, Inc)