Meeting 2:

Materials course: **basic properties of ring**

Theorem 1. If R is ring, then for every $a,b,c \in R$ the following statements are satisfied.

1. $az = za = z$
2. $a(-b) = (-a)b = -(ab)$
3. $(-a)(-b) = ab$
4. $-(a+b) = (-a)+(-b)$
5. $a(b-c) = ab - ac$
6. $(a-b)c = ac - bc$
7. $(-u)a = -a$ where *u* is unity

Theorem 2. Let R be ring. The ring R has no zero divisor if and only if the canselation law is satisfied in R.

Theorem 3. The finite integral domain is field.

Exercises:

1. Proof Theorem 1, 2 and 3.

Definition 1.

Let R be ring and $a \in R$ and m be a positive integer. We have the following definitions.

1. ma = $\underbrace{a+a+a+...+a}_{m}$
2. $-ma = \underbrace{(-a)+(-a)+. \quad +.(-a)}_{m} = -(ma)$
3. $0a = z$

Theorem 4. If R is ring and m, n are integers, then

1. $(m+n)a = ma + na$
2. $m(a+b) = ma + mb$
3. $m(na) = (mn)a = n(ma)$

Definition 2. If R is ring, $a \in R$ and m is a positive integer, then we define

$a^m = \underbrace{a.a.a...a}_{m}$

If $a \in R$ and $a^2 = a$, then a is called idempotent element of R.

If $a \in R$ and there exists a positive integer n such that $a^n = z$, then a is called nilpotent element of R.

Definition 3. Let R be ring. If there exists the least positive integer n such that $na = z$ for every $a \in R$, then n is called **characteristic of ring R** . If there no such number, then the **characteristic of R** is 0 or infinite.

Examples.

1. Identify characteristic of ring $\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}$.
2. Find characteristic of $n\mathbb{Z}$ and $\mathbb{Z}_n$.

Theorem 5. Let $(R,+,\times)$ be an integral domain and $a,b\in R, a\neq z, b\neq z$, then p(a) = p(b) (period of a is equal to period of b under group (R,+).

Theorem 6. Let $(R,+,\times)$ be an integral domain, then characteristic of R is 0 or a positive integer *n* such that *n* is period of nonzeero element of R under group (R,+).

Theorem 7. Let $(R,+,\times)$ be an integral domain, then characteristic of R is 0 or a prime number.

Exercises:

Find the characteristic of the given ring:

1. $2\mathbb{Z}$
2. $\mathbb{Z}\times\mathbb{Z}$
3. $\mathbb{Z}_3\times 3\mathbb{Z}$
4. $\mathbb{Z}_3\times\mathbb{Z}_3$
5. $\mathbb{Z}_3\times\mathbb{Z}_4$
6. $\mathbb{Z}_6\times\mathbb{Z}_{15}$
7. Let R be commutative ring with unity of characteristic 3. Compute and simplify $(a+b)^4$ for $a,b\in R$.
8. Let R be commutative ring with unity of characteristic 3. Compute and simplify $(a+b)^3$ for $a,b\in R$.
9. Let R be commutative ring with unity of characteristic p where p is prime number. Compute and simplify $(a+b)^p$ for $a,b\in R$.
10. Let R be an integral domain of order m, show that characteristic of R divides m.
11. Let F be a field of order 8, find characteristic of F.
12. Let F be a field of order 2^n, find characteristic of F.
13. Let F be a field of order p^n where p is prime number, find characteristic of F.

Let R be ring and $a\in R$. If there exists positive integer n such that $a^n = z$, then a is called a **nilpotent element**. If $a^2 = a$, then a is called an **idempotent element**.

14. Find all nilpotent elements of integral domain R.
15. Find all idempotent elements of integral domain R.
16. Determine all ring elements that are both nilpotents and idempotents.
17. Let R be ring with unity and $a\in R$. Suppose $a^n = z$ for some positive integer n, prove that $1-a$ has multiplicative inverse in R. (hint: compute $(1-a)(1+a+a^2+\ldots+a^{n-1})$).