Regions in the Complex Plane

The concept of an ε *neighborhood* of a given point z_0, it is written $N(z_0,\varepsilon)=\{z_0\in\mathbb{C}:|z-z_0|<\varepsilon\}$.

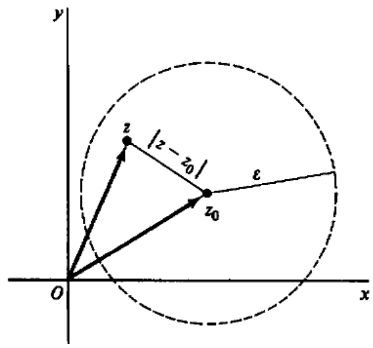

Figure 4. An ε *neighborhood*

When it consists of all points z in an ε *neighborhood* of z_0 except for the point z_0 itself, it is called a *deleted neighborhood*. Some definitions of regions in the complex plane are written below:

(1) A point z_0 is said to be an *interior point* of a set *E* if there exist a neighborhood of z_0 that contains only points of *E*. It is written: $\exists\varepsilon>0\ \forall z\in\mathbb{C}$ so that $N(z_0,\varepsilon)\subseteq E$

(2) A point z_0 is said to be an *exterior point* of a set *E* if there exist a neighborhood of z_0 that contains only points of E^C. It is written: $\exists\varepsilon>0\ \forall z\in\mathbb{C}$ so that $N(z_0,\varepsilon)\subseteq E^C$.

(3) A point z_0 is said to be an *boundary point* of a set *E* if it is neither interior point nor exterior points of *E*. It is written: $\forall\varepsilon>0\ \forall z\in\mathbb{C}$ so that $N(z_0,\varepsilon)\cap E\neq\varnothing$ and $N(z_0,\varepsilon)\cap E^C\neq\varnothing$

(4) A point z_0 is said to be an *accumulation point (limit point)* of a set *E* if each deleted neighborhood of z_0 contains at least one point of *E*. It is written: $\forall\varepsilon>0\ \forall z\in\mathbb{C}$ so that $N(z_0,\varepsilon)\cap E-z_0\neq\varnothing$

(5) A point z_0 is said to be an *isolated point* of a set *E* if there exists a *neighborhood* of z_0 not containing other points of *E*. Equivalently, a point z_0 in *E* is an isolated point of *E* if and only if it is not a limit point of *E*. It is written: $\exists \varepsilon > 0\ \forall z \in \mathbb{C}$ so that $N(z_0, \varepsilon) \cap E = \varnothing$

(6) A set is *open* if it contains all of its interior points.

(7) A set is *closed* if it contains all of its limit points.

(8) An open set *E* is *connected* if each pair of points z_1 and z_2 in it can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in *E*.

(9) An open set that is connected is called a *domain*.

(10) A domain together with some, none, or all of its boundary points is referred to as a *region*.

(11) A set *E* is said to be bounded if every point of *E* lies inside some circle $|z| = M$. It is written: $\exists M > 0$ so that $|z| < M$

Exercises:

1. Sketch the following sets then determine all interior points, exterior points, limit points, isolated points and boundary points

   (a) $\{z \in \mathbb{C} : |z - 1 + 1| < 1\}$

   (b) $\{z \in \mathbb{C} : \mathrm{Im}(z) > 2\}$

   (c) $\{z \in \mathbb{C} : |z - 3i| + |z + 3i| \leq 10\}$

   (d) $\{z \in \mathbb{C} : \mathrm{Re}(z) \leq 1\}$

   (e) $\{z \in \mathbb{C} : 0 < |z| < 1\} \cup \{1 + i\}$

   (f) $\{z \in \mathbb{C} : 1 < |z - 2 + i| < 3\}$

2. Which sets in Exercises 3 are domains?
3. Which sets in Exercises 3 are bounded?
4. Which sets in Exercises 3 are open set?