

Regions in the Complex Plane

The concept of an ε neighborhood of a given point z_0 , it is written

$$N(z_0, \varepsilon) = \{z_0 \in \mathbb{C} : |z - z_0| < \varepsilon\}.$$

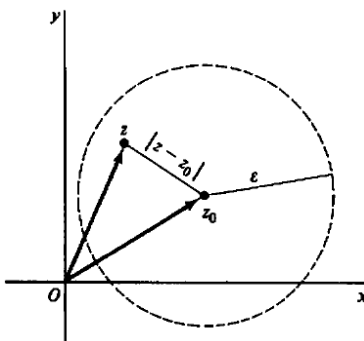


Figure 4. An ε neighborhood

When it consists of all points z in an ε neighborhood of z_0 except for the point z_0 itself, it is called a *deleted neighborhood*. Some definitions of regions in the complex plane are written below:

- (1) A point z_0 is said to be an *interior point* of a set E if there exist a neighborhood of z_0 that contains only points of E . It is written: $\exists \varepsilon > 0 \forall z \in \mathbb{C}$ so that $N(z_0, \varepsilon) \subseteq E$
- (2) A point z_0 is said to be an *exterior point* of a set E if there exist a neighborhood of z_0 that contains only points of E^c . It is written: $\exists \varepsilon > 0 \forall z \in \mathbb{C}$ so that $N(z_0, \varepsilon) \subseteq E^c$.
- (3) A point z_0 is said to be an *boundary point* of a set E if it is neither interior point nor exterior points of E . It is written: $\forall \varepsilon > 0 \forall z \in \mathbb{C}$ so that $N(z_0, \varepsilon) \cap E \neq \emptyset$ and $N(z_0, \varepsilon) \cap E^c \neq \emptyset$
- (4) A point z_0 is said to be an *accumulation point (limit point)* of a set E if each deleted neighborhood of z_0 contains at least one point of E . It is written: $\forall \varepsilon > 0 \forall z \in \mathbb{C}$ so that $N(z_0, \varepsilon) \cap E - z_0 \neq \emptyset$

- (5) A point z_0 is said to be an *isolated point* of a set E if there exists a *neighborhood* of z_0 not containing other points of E . Equivalently, a point z_0 in E is an isolated point of E if and only if it is not a limit point of E . It is written: $\exists \varepsilon > 0 \forall z \in \mathbb{C}$ so that $N(z_0, \varepsilon) \cap E = \emptyset$
- (6) A set is *open* if it contains all of its interior points.
- (7) A set is *closed* if it contains all of its limit points.
- (8) An open set E is *connected* if each pair of points z_1 and z_2 in it can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in E .
- (9) An open set that is connected is called a *domain*.
- (10) A domain together with some, none, or all of its boundary points is referred to as a *region*.
- (11) A set E is said to be bounded if every point of E lies inside some circle $|z| = M$. It is written:
- $$\exists M > 0 \text{ so that } |z| < M$$

Exercises:

- Sketch the following sets then determine all interior points, exterior points, limit points, isolated points and boundary points

(a) $\{z \in \mathbb{C} : z - 1 + i < 1\}$	(d) $\{z \in \mathbb{C} : \operatorname{Re}(z) \leq 1\}$
(b) $\{z \in \mathbb{C} : \operatorname{Im}(z) > 2\}$	(e) $\{z \in \mathbb{C} : 0 < z < 1\} \cup \{1 + i\}$
(c) $\{z \in \mathbb{C} : z - 3i + z + 3i \leq 10\}$	(f) $\{z \in \mathbb{C} : 1 < z - 2 + i < 3\}$
- Which sets in Exercises 3 are domains?
- Which sets in Exercises 3 are bounded?
- Which sets in Exercises 3 are open set?