MEETING: 1-2

Learning Objectives: Students are able to

1. Explain complex number system
2. Explain algebraic properties of complex number

1. Definition of Complex Numbers

Complex numbers can be defined as ordered pairs (x, y) of real numbers. Let $i = \sqrt{-1}$, so $i^2 = -1$ be some fixed symbol (we shall call it "imaginary unit"). An expression $z = x + iy$ $(x, y \in \mathbb{R})$ is called a *complex number*. We denote a set of complex numbers by $\mathbb{C}$. Then set $\mathbb{C}$ of all complex numbers is defined by

$$\mathbb{C} = \left\{ z = x + iy \middle| x, y \in \mathbb{R} \text{ and } i = \sqrt{-1} \right\}$$

The real numbers x and y are known as the *real and imaginary parts* of z ; and we write

$$\text{Re } z = x, \qquad \text{Im } z = y$$

2. Algebraic Properties

Various properties of addition and multiplication of complex numbers are the same as for real numbers.

(1) The commutative laws

$$z_1 + z_2 = z_2 + z_1$$
$$z_1 z_2 = z_2 z_1$$

(2) The associative laws

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

(3) Distributive laws

$$z(z_1 + z_2) = zz_1 + zz_2$$

Exercises

1. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when

(a) $z_1 = 1 + \frac{1}{2}i,\ z_2 = \frac{2}{3}i$

(b) $z_1 = 2 - \sqrt{3}i,\ z_2 = \frac{1}{2} + i$

(c) $z_1 = x_1 - x_2 i,\ z_2 = \frac{1}{2}x_1 + x_2 i$

2. Find $\operatorname{Re}(z), \operatorname{Im}(z), |z|, \bar{z}$ if

(a) $z = \dfrac{3-4i}{2+3i} + \dfrac{2}{i}$ (b) $z = \dfrac{i^4 - i^9 + i^3}{(1+i)(1-2i)}$

MEETING: 3-4

Learning Objectives: Students are able to Determine modulli and conjugate of complex number

Modulli and Conjugate

The *modulus* or absolute value of the complex number $z = x + iy$ is defined as the nonnegative real number and denoted by $|z|$; that is $|z| = \sqrt{x^2 + y^2}$.

The *complex conjugate* of a complex number $z = x + iy$ is defined as the complex number $x - iy$ and denoted by $\bar{z}$; that is $\bar{z} = x - iy$.

Example 1

$$\frac{1+3i}{2-i} = \frac{1+3i}{2-i}\,\frac{2+i}{2+i} = \frac{-1+7i}{|2-i|^2} = \frac{-1+7i}{5} = -\frac{1}{5} + \frac{7}{5}i$$

Example 2

If z is a point inside the circle centered at the origin and with radius 3, so that $|z| < 3$, then

$$|z^3 + 2z^2 - z + 1| \le |z|^3 + 2|z|^2 - |z| + 1 < 3^3 + 2(3)^2 - 3 + 1 = 43.$$

Exercises:

1. Suppose $z_1 = 1 + i$ and $z_2 = 2 + i$. Evaluate of each following

   (a) $|3z_1 - 4z_2|$ (c) $(\bar{z}_2)^4$

   (b) $z_1^3 - 3z_2^2 + 2z_1 - 4$ (d) $\left|\dfrac{3z_2 - 4 - i}{2z_1}\right|^2$

2. Find real numbers x and y such that $3x + 2iy - ix + 5y = 7 + 5i$.

3. If $z_1, z_2 \ne 0$, then show that $\overline{\left(\dfrac{z_1}{z_2 z_3}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2 \bar{z}_3}$.

4. If $|z| = 3$, then show that $\left|\dfrac{1}{z^2+1}\right| \le \dfrac{1}{6}$.