

## MEETING: 21-22

Learning Objectives: Students are able to

1. explain hyperbolic functions
2. explain logarithmic functions

### 1. The Hyperbolic Function

**Definition:**

$$\forall z \in \mathbb{C}, \sinh z = \frac{e^z - e^{-z}}{2}, \cosh z = \frac{e^z + e^{-z}}{2}.$$

Furthermore,

$$\tanh z = \frac{\sinh z}{\cosh z},$$

$$\coth z = \frac{\cosh z}{\sinh z},$$

$$\operatorname{sech} z = \frac{1}{\cosh z},$$

$$\operatorname{csch} z = \frac{1}{\sinh z},$$

$$\frac{d}{dz} \sinh z = \cosh z,$$

$$\frac{d}{dz} \cosh z = \sinh z,$$

$$\frac{d}{dz} (\tanh z) = \operatorname{sech}^2 z,$$

$$\frac{d}{dz} (\operatorname{sech} z) = -\operatorname{sech} z \tanh z.$$

The relationship between hyperbolic function and trigonometric function:

- a.  $\cosh(iz) = \cos z$
- b.  $\cos(iz) = \cosh z$
- c.  $-i \sinh(iz) = \sin z$
- d.  $-i \sin(iz) = \sinh z$

Some of the most frequently used identities involving hyperbolic sine and cosine functions are

$$\sinh(-z) = -\sinh z,$$

$$\cosh(-z) = \cosh z,$$

$$\cosh^2 z - \sinh^2 z = 1,$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2,$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.$$

$$\sinh z = \sinh x \cos y + i \cosh x \sin y, \quad \forall z = x + iy,$$

$$\cosh z = \cosh x \cos y + i \sinh x \sin y, \quad \forall z = x + iy,$$

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y, \quad \forall z = x + iy,$$

$$|\cosh z|^2 = \cosh^2 x + \cos^2 y, \quad \forall z = x + iy,$$

### 2. The Logarithmic Function

For every  $z = re^{i\theta}$ ,

$$\log z = \ln r + i(\theta + 2n\pi), n \in \mathbb{Z}.$$

The principal value of  $\log z$  is denoted by  $\text{Log } z$ , define  $\text{Log } z = \ln r + i\theta, -\pi < \theta \leq \pi$ .

*Example*

$$\log(1+i) = \ln \sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right), n = 0, \pm 1, \pm 2, \dots \quad \text{and} \quad \text{the} \quad \text{principal} \quad \text{value} \quad \text{is}$$

$$\text{Log}(1+i) = \ln \sqrt{2} + i\left(\frac{\pi}{4}\right).$$

**Properties:**

Let  $z, z_1, z_2 \in \mathbb{C}$

a.  $\frac{d}{dz} \log z = \frac{1}{z}$

c.  $\log(z_1 z_2) = \log z_1 + \log z_2$

b.  $e^{\log z} = z$

d.  $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$

**Exercises:**

1. Show that  $\sin(iz) = i \sinh z$
2. Find all values of  $z$  for which the following equations hold
  - a.  $\sinh z = \frac{i}{2}$
  - b.  $\cosh z = 1$
3. Show that  $\sinh(z + i\pi) = -\sinh z$
4. Find the principal value for the following
  - a.  $\text{Log}(ie^2)$
  - b.  $\text{Log}(i\sqrt{2} - \sqrt{2})$
5. Find all the values of  $\log z$  for the  $\log(-3)$
6. Find all the values of  $z$  for which the following equations hold:
  - a.  $\text{Log } z = 1 - \frac{i\pi}{4}$
  - b.  $e^z = -ie$