1. Functions of a Complex Variable

Let E be a set of complex numbers. A *function f* defined on E is a rule that assigns to each z in E a complex number w. The number w is called the *value* of f at z and is denoted by $f(z)$; that is, $w = f(z)$. The set E is called the *domain of definition of f*, written $D(f)$, $D(f) = \{z \in \mathbb{C} : f(z) \ \textit{defined}\}$.

Example 1

Determine the domain definition of $f(z) = \dfrac{z+1}{z^2+z+1}$ and $g(z) = z^2 + z + 1$

Solution:

$$
\begin{aligned}
D(f) &= \{z \in \mathbb{C} : f(z) \ \textit{defined}\} \\
&= \{z \in \mathbb{C} : z^2 + z + 1 \neq 0\} \\
&= \left\{z \in \mathbb{C} : z \neq -\frac{1}{2} \pm \frac{1}{2} i\sqrt{3}\right\}
\end{aligned}
$$

$$
\begin{aligned}
D(g) &= \{z \in \mathbb{C} : g(z) \ \textit{defined}\} \\
&= \mathbb{C}
\end{aligned}
$$

2. Limits

Let a function f be defined at all points z in some deleted neighborhood of z_0. The statement that the *limit* of $f(z)$ as z approaches z_0 is a number w_0, or

$$\lim_{z \to z_0} f(z) = w_0 \tag{2}$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it. Equation (2) means that, for each positive number ε, there is a positive number δ such that

$$|f(z) - w_0| < \varepsilon \qquad \text{whenever} \qquad 0 < |z - z_0| < \delta . \tag{3}$$

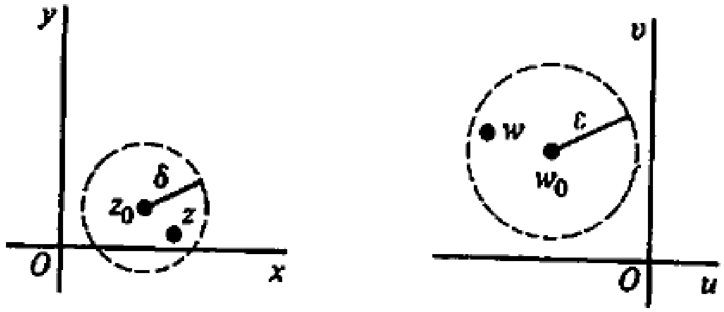

Figure 4. Limit function

Example 1

Show that $\lim_{z\to i} z^2-1=2$

Solution:

From the properties of modulli, we have

$$\left|\left(z^2-1\right)-(-2)\right|=\left|z^2+1\right|=|z-i||z+i|.$$

Observe that when $\forall z\in\mathbb{C}$ is in the region $|z-i|<1$,

$$|z+i|=|z-i+2i|\le|z-i|+|2i|<1+2=3.$$

Hence, for $\forall z\in\mathbb{C}$ such that $|z-i|<1$,

$$\left|\left(z^2-1\right)-(-2)\right|=\left|z^2+1\right|=|z-i||z+i|<3|z-i|.$$

For any positive number ε, get $\delta=\min\left\{\frac{\varepsilon}{3},1\right\}$ such that $0<|z-i|<\delta$,

$$\left|\left(z^2-1\right)-(-2)\right|<3|z-i|<3\left(\frac{\varepsilon}{3}\right)=\varepsilon. \qquad \blacksquare$$

Note that when a limit function $f(z)$ exist at a point z_0, it is unique.

Example 2

If $f(z)=\frac{\overline{z}}{z}$, then $\lim_{z\to 0} f(z)$ does not exist.

When $z=(x,0)$ is a nonzero point on the real axis,

$$f(z)=\frac{x-i0}{x+i0}=1$$

and when $z=(0,y)$ is a nonzero point on the imaginary axis,

$$f(z)=\frac{0-iy}{0+iy}=-1.$$

Thus, by letting z approach the origin along real axis, we would find that the desired limit is 1. An approach along imaginary axis would, on the other hand, yield the limit -1. Since the limit is unique, we must conclude that $\lim_{z\to 0}\frac{\overline{z}}{z}$ does not exist.

Since limits of the latter type are studied in calculus, we use their definition and properties freely.

Theorem 1

Suppose that

$f(z)=u(x,y)+iv(x,y),\qquad z_0=x_0+iy_0,\quad$ *and* $\qquad w_0=u_0+iv_0.$

Then

$$\lim_{z\to z_0} f(z)=w_0 \tag{4}$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y)=u_0 \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} v(x,y)=v_0 \tag{5}$$

Example

Find $\lim_{z\to 1+i}\left(z^2+\frac{1}{z}\right)$.

Observe that $z^2+\frac{1}{z}=(x+iy)^2+\frac{1}{x+iy}=\left(x^2-y^2\right)+\frac{x}{x^2+y^2}+i\left(2xy-\frac{y}{x^2+y^2}\right)$.

We have $u(x,y)=\left(x^2-y^2\right)+\frac{x}{x^2+y^2}$ and $v(x,y)=2xy-\frac{y}{x^2+y^2}$. From Theorem 1,

$$\lim_{(x,y)\to(1,i)}\left[\left(x^2-y^2\right)+\frac{x}{x^2+y^2}\right]=\frac{1}{2} \qquad \text{and} \qquad \lim_{(x,y)\to(1,i)}\left[2xy-\frac{y}{x^2+y^2}\right]=\frac{3}{2},$$

thus

$$\lim_{z\to 1+i}\left(z^2+\frac{1}{z}\right)=\frac{1}{2}+\frac{3}{2}i.$$

Theorem 2

If $\lim_{z\to z_0} f(z)$, $\lim_{z\to z_0} g(z)$ *exist and* $c\in\mathbb{C}$, *then*

(1) $\lim_{z \to z_0} (f(z) + g(z))$ *exist and* $\lim_{z \to z_0} (f(z) + g(z)) = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$

(2) $\lim_{z \to z_0} (cf(z))$ *exist and* $\lim_{z \to z_0} (cf(z)) = c \lim_{z \to z_0} f(z)$

(3) $\lim_{z \to z_0} (f(z) g(z))$ *exist and* $\lim_{z \to z_0} (f(z) g(z)) = \lim_{z \to z_0} f(z) \lim_{z \to z_0} g(z)$

(4) $\lim_{z \to z_0} \left(\frac{f(z)}{g(z)} \right)$ *exist and* $\lim_{z \to z_0} \left(\frac{f(z)}{g(z)} \right) = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}$, *whenever* $\lim_{z \to z_0} g(z) \neq 0$

Limits Involving the Point at Infinity

We have three point about limits that is involving the point at infinity:

$$\lim_{z \to z_0} f(z) = \infty \qquad \text{if and only if} \qquad \lim_{z \to z_0} \frac{1}{f(z)} = 0. \tag{6}$$

$$\lim_{z \to \infty} f(z) = w_0 \qquad \text{if and only if} \qquad \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0. \tag{7}$$

$$\lim_{z \to \infty} f(z) = \infty \qquad \text{if and only if} \qquad \lim_{z \to 0} \frac{1}{f\left(1/z\right)} = 0. \tag{8}$$

Example 1

Observe that $\lim_{z \to -i} \frac{i+3}{z+i} = \infty$ since $\lim_{z \to -i} \frac{z+i}{i+3} = 0$.

Example 2

Observe that $\lim_{z \to \infty} \frac{3z-i}{z+2} = 3$ since $\lim_{z \to 0} \frac{3\left(1/z\right) - i}{\left(1/z\right) + 2} = \lim_{z \to 0} \frac{3 - iz}{1 + 2z} = 3$.

Example 3

Observe that $\lim_{z \to \infty} \frac{3z^4 - i}{z^3 + 2} = \infty$ since $\lim_{z \to 0} \frac{\left(1/z^3\right) + 2}{3\left(1/z^4\right) - i} = \lim_{z \to 0} \frac{z + 2z^4}{3 - iz^4} = 0$.

Exercises

1. For each of the functions below, describe the domain of definition that is understood

(a) $f(z) = \frac{1}{z^2 + 4}$ (c) $f(z) = \cos(z^2 - i)$

(b) $f(z)=\dfrac{\overline{z}+2i}{z+\overline{z}}$

2. Write the function $f(z)=z^3+2z-i$ in the form $f(z)=u(x,y)+iv(x,y)$.

3. Let z_0, c denote complex constant. Use definition (3) to prove that

(a) $\lim\limits_{z\to z_0} c=c$ (b) $\lim\limits_{z\to 1-i}(x+i2y)=1-2i$

4. Let $f(z)=\dfrac{z^2}{|z|^2}$

a. Find $\lim\limits_{z\to 0} f(z)$ along the line $y=x$

b. Find $\lim\limits_{z\to 0} f(z)$ along the line $y=2x$

c. Find $\lim\limits_{z\to 0} f(z)$ along the parabola $y=x^2$

d. What can you conclude about the limit of $f(z)$ along $z\to 0$

5. Using (6), (7) and (8) of limits, show that

(a) $\lim\limits_{z\to\infty}\dfrac{z^4-z^3+2z}{(z+1)^4}=1$ (c) $\lim\limits_{z\to\infty}\dfrac{z^2-1}{z+1}=\infty$

(b) $\lim\limits_{z\to 2i}\dfrac{z}{(z-2i)^2}=\infty$

6. Find the value of limits below

(a) $\lim\limits_{z\to 1+2i} z^2+2z-1$ (c) $\lim\limits_{z\to(1+i\sqrt{3})}\dfrac{z^3+8}{z^4+4z^2+16}$ (e) $\lim\limits_{z\to i}\dfrac{z^2+4z+2}{z+1}$

(b) $\lim\limits_{z\to -2i}\dfrac{z^2+2z-1}{z^2-2z+4}$ (d) $\lim\limits_{z\to i}\dfrac{z^4-1}{z-i}$ (f) $\lim\limits_{z\to 1+i}\dfrac{z^2+z-1-3i}{z^2-2z+2}$