MEETING: 11

Learning Objectives: Students are able to determine continuity of complex variable

Continuity

A function *f* is *continuous* at point z_0 if all three of the following conditions are satisfied

$$\lim_{z \to z_0} f(z) \qquad \text{exists,} \tag{9}$$

$$f(z_0) \text{ exist,} \tag{10}$$

$$\lim_{z \to z_0} f(z) = f(z_0). \tag{11}$$

Statement (11) says that, for each positive number ε, there is a positive number δ such that

$$|f(z) - f(z_0)| < \varepsilon \qquad \text{whenever} \qquad |z - z_0| < \delta. \tag{12}$$

A function of a complex variable is said to be continuous in a region *R* if it is continuous at each point in *R*.

Theorem

If $f(z), g(z)$ *are continuous at* $z = z_0$, *then*

(1) $f(z) \pm g(z)$

(2) $f(z)g(z)$

(3) $\dfrac{f(z)}{g(z)}$, $g(z_0) \neq 0$

are also continuous at $z = z_0$.

Note that, a polynomial, natural exponential and trigonometric function are continuous in the entire plane. It follows directly from definition (12) that *a composition of continuous functions is continuous.*

Example 1

The function

$f(z) = z^2 + 3z - i$ is continuous everywhere in the complex plane since it polynomial function.

Example 2

Suppose

$$f(z)=\begin{cases} z^2, & z\neq i \\ 0, & z=i \end{cases}$$

Then, $\lim_{z\to i} f(z)=-1$, but $f(i)=0$. Hence $\lim_{z\to i} f(z)\neq f(i)$ and the function is not continuous at $z=i$.

Points in the complex plane where $f(z)$ fails to be continuous are called discontinuities of $f(z)$, and $f(z)$ is said to be *discontinuous* at these points. If $\lim_{z\to z_0} f(z)$ exist but is not equal to $f(z_0)$, we call z_0 a *removable discontinuity* since by redefining $f(z_0)$ to be same as $\lim_{z\to z_0} f(z)$, the function becomes continuous.

Exercises

1. Is the function $f(z)=\dfrac{z^3+2z^2-3z-1}{z-3i}$ continuous at $z=3i$? If not, is it can be continuous? How?
2. For what values of z are each of the following function continuous?
   (a) $f(z)=\dfrac{3i}{z^2+3}$
   (b) $f(z)=\dfrac{1}{\cos z}$