MEETING: 14-15

Learning Objectives: Students are able to

1. explain Cauchy-Riemann equations
2. explain sufficient conditions for differentiability

1. Cauchy-Riemann Equations

In this section, we obtain a pair of equations that the first-order partial derivative of the component functions *u* and *v* of a function

$$f(z) = u(x, y) + iv(x, y) \tag{18}$$

must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of *f* exist there. We also show how to write $f'(z_0)$ in terms of those partial derivative.

Theorem

Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = (x_0, y_0)$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy the Cauchy-Riemann equations

$$u_x = v_y \qquad \textit{and} \qquad u_y = -v_x \tag{19}$$

there. Also, $f'(z_0)$ can be written

$$f'(z_0) = u_x + iv_x \tag{20}$$

where these partial derivatives are to be evaluated at (x_0, y_0).

Example

The function $f(z) = z^2 + 3 - i = x^2 - y^2 + 3 + i(2xy - 1)$ is differential everywhere and that $f'(z) = 2z$. To verify that Cauchy-Riemann equations are satisfied everywhere, we note that $u(x, y) = x^2 - y^2 + 3$ and $v(x, y) = 2xy - 1$. Thus $u_x = 2x = v_y$ and $u_y = -2y = -v_x$. Moreover, according to equation (20), $f'(z) = 2x + i2y = 2(x + iy) = 2z$.

2. Sufficient Conditions for Differentiability

Theorem

Let the function $f(z)=u(x,y)+iv(x,y)$ *be defined throughout some* ε *neighborhood of a point* $z_0=x_0+iy_0$. *Suppose that the first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in that neighborhood and that they are continuous at* (x_0,y_0). *Then, if those partial derivatives satisfy the Cauchy-Riemann equations* $u_x=v_y$ *and* $u_y=-v_x$ *at* (x_0,y_0), *the derivative* $f'(z_0)$ *exists.*

Exercises

1. Use theorem in Sec.1 to show that $f'(z)$ does not exist at any point if

   a. $f(z)=\overline{z}$ b. $f(z)=z+\overline{z}$ c. $f(z)=x+ixy$ d. $f(z)=e^x(\cos y-i\sin y)$

2. Use theorem in Sec.2 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere and find $f''(z)$ when

   a. $f(z)=iz+2011$ b. $f(z)=z^2$

3. Find the constant *a* and *b* such that $f(z)=(2x-y)+i(ax+by)$ is differentiable for all *z*.

4. Let $f(z)=|z|^2$. Show that *f* is differentiable at a point $z_0=0$ but is not differentiable at any other point.