Cusp-Bodanov-Takens Bifurcation in a Predator-Prey Type of Dynamical System with Time-Periodic Perturbation

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Outline

- Predator-Prey System
- Cusp-Bogdanov-Takens Bifurcation
- System with Time-Periodic Perturbation
- Conclusion
- References

Predator-Prey System

Given Predator-Prey system

$$\begin{aligned} \dot{x} &= ax - \lambda x^2 - yP(x) \\ \dot{y} &= -\delta y - \mu y^2 + yQ(x) \end{aligned} \quad (1)$$

where $a, \lambda, \delta > 0$ dan $\mu \geq 0$,

$$\begin{aligned} P(x) &= \frac{mx}{\alpha x^2 + \beta x + 1} \\ Q(x) &= cP(x) \end{aligned}$$

where $\alpha \geq 0$ dan $m, c > 0$ dan $\beta > -2\sqrt{\alpha}$.

Related works

Broer, et al., *Bifurcation of a predator-prey model with non-monotonic response function*, Comptes Rendus Mathematique, V. 341, (2005) 601-604.

Broer, et al., *A predator-prey model with non-monotonic response function*, Regular and Chaotic Dynamics 11(2) (2006),155-165.

Broer, et al., *Dynamics of a predator-prey model with non-monotonic response function*, DCDS-A 18(2-3) (2007), 221-251.

Haryanto, E., Tuwankotta, J.M., *Swallowtail in predator-prey type of system with time-periodic perturbation*, submitted to ICREM 2011.

Tuwankotta, J.M., Haryanto, E., *On periodic solution of a predator-prey type of dynamical system with time-periodic perturbation*, submitted to ICREM 2011.

Our goal:

Show that there is a codimension 3 bifurcation (Cusp-Bogdanov-Takens) on the system.

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We give time perturbation on the carrying capacity to see its influence to the bifurcation
the variation is $\lambda = \lambda_0(1 + \epsilon \sin(\omega t))$.

Scaling

Define:

$$
\begin{aligned}
t &= \frac{1}{a}\tau, \quad x = \frac{a}{cm}\widetilde{x}, \quad y = \frac{a}{m}\widetilde{y}, \qquad \lambda = cm\widetilde{\lambda}, \\
\delta &= a\widetilde{\delta}, \quad \mu = m\widetilde{\mu}, \qquad \alpha = \frac{c^2m^2}{a^2}\widetilde{\alpha}, \quad \beta = \frac{cm}{a}\widetilde{\beta}
\end{aligned}
$$

substitute to the system (1) and discard the tilde on the new variable:

$$
\begin{aligned}
\dot{x} &= x(1 - \lambda x - \frac{y}{\alpha x^2 + \beta x + 1}) \\
\dot{y} &= y(-\delta - \mu y + \frac{y}{\alpha x^2 + \beta x + 1}).
\end{aligned} \tag{2}
$$

We fix the value $\lambda = 0.01$ and $\delta = 1.1$.

Equilibrium points

The system (2) has some equilibrium points:

1. $O(0,0)$, where $\lambda_1 = 1$ and $\lambda_2 = -1.1$.
2. $(100,0)$, $\lambda_1 = -1$ and $\lambda_2 = -1.1 + \frac{0.01}{0.0001+0.01\beta+\alpha}$.
3. $(0, -\frac{\delta}{\mu})$, for $\mu \neq 0$.
4. for $x \neq 0$ and $y \neq 0$, the equilibrium point of the system (2) is the solution of

$$\begin{aligned} 1 - 0.01x - \frac{y}{\alpha x^2 + \beta x + 1} &= 0 \\ -1.1 - \mu y + \frac{x}{\alpha x^2 + \beta x + 1} &= 0 \end{aligned}$$

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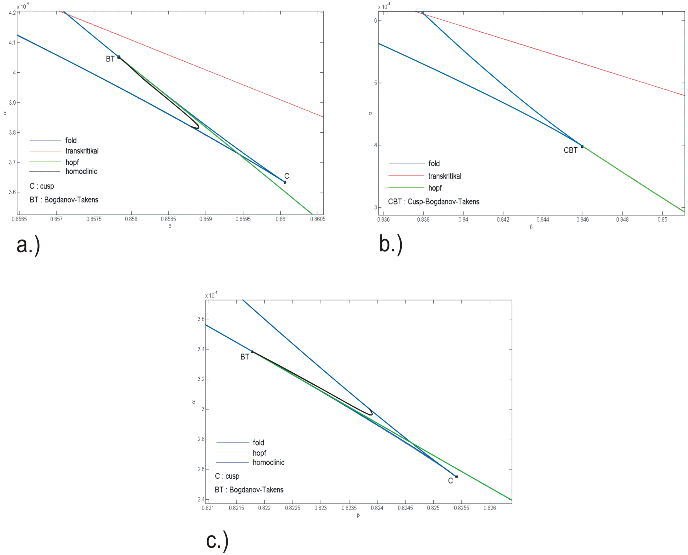
The Cusp-Bogdanov-Takens bifurcation point is $(\bar{\alpha}, \bar{\beta}, \bar{\mu}) = (0.0003986797, 0.8459684, 0.0013)$ at $(\bar{x}, \bar{y}) = (49.38394909, 22.14429463)$.

Cusp-Bogdanov-Takens Bifurcation



Cusp-Bogdanov-Takens Bifurcation Diagram

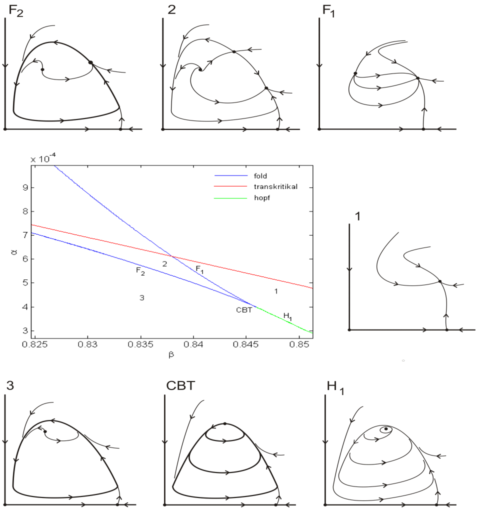


Figure: Cusp-Bogdanov-Takens Bifurcation Diagram and its Phase Portrait

System with Periodic Perturbation

Before the periodic perturbation, the system is:

$$\begin{aligned} \dot{x} &= x(1 - \lambda x - \frac{y}{\alpha x^2 + \beta x + 1}) \\ \dot{y} &= y(-\delta - \mu y + \frac{y}{\alpha x^2 + \beta x + 1}). \end{aligned} \tag{3}$$

Parameter λ, is varied by

$$\lambda = \lambda_0 \left(1 + \epsilon \sin\left(\omega t\right)\right) \tag{4}$$

where λ_0, ω are constants and $\epsilon > 0$ is the perturbation parameter. Thus we get

$$\begin{aligned} \dot{x} &= x - \lambda x^2 - \lambda \epsilon \sin(\omega t) x^2 - \frac{xy}{\alpha x^2 + \beta x + 1} \\ \dot{y} &= -\delta y - \mu y^2 + \frac{xy}{\alpha x^2 + \beta x + 1} \end{aligned} \tag{5}$$

the strategy

Define

$$\begin{aligned} \dot{u} &= \omega v + u - u(u^2+v^2) \\ \dot{v} &= -\omega u + v - v(u^2+v^2). \end{aligned} \qquad (6)$$

System (6) has solutions $u = \sin \omega t$ and $v = \cos \omega t$.

$$\begin{aligned} \dot{u} &= \omega v + u - u(u^2+v^2) \\ \dot{v} &= -\omega u + v - v(u^2+v^2) \\ \dot{x} &= x - \lambda x^2 - \lambda \varepsilon u x^2 - \frac{xy}{\alpha x^2 + \beta x + 1} \\ \dot{y} &= -\delta y - \mu y^2 + \frac{xy}{\alpha x^2 + \beta x + 1} \end{aligned}$$

We investigate the system at $\omega = 1$, $\lambda = 0.01$, and $\delta = 1.1$ for the perturbations $\epsilon = 0.0003$, $\epsilon = 0.001$, and $\epsilon = 0.02$.

Bifurcation Diagrams after the Perturbations

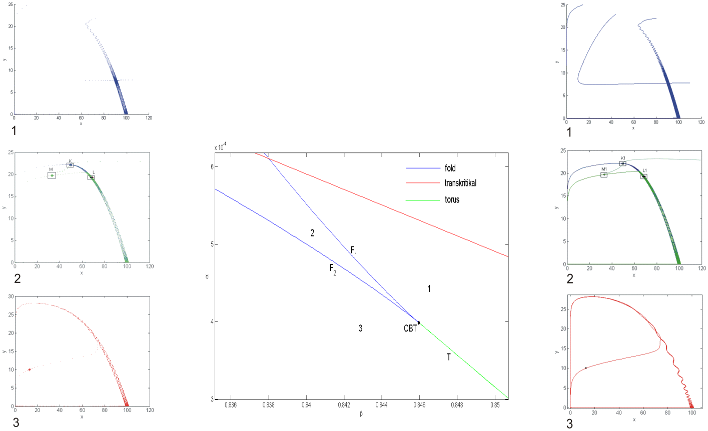


Figure: Bifurcation diagram of the system (5) for $\epsilon > 0$

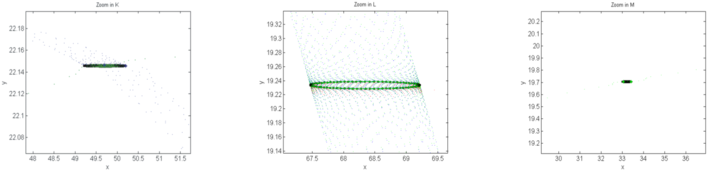


Figure: Zoom in K (left), L (middle), dan M (right)

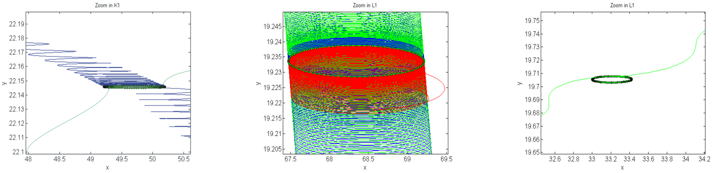


Figure: Zoom in K1 (left), L1 (middle), dan M1 (right)

Conclusion

1. For the value $\lambda = 0.01$ and $\delta = 1.1$ we get that the Cusp-Bogdanov-Takens bifurcation happens at $(\bar{\alpha}, \bar{\beta}, \bar{\mu}) = (0.0003986797, 0.8459684, 0.0013)$ for $(\bar{x}, \bar{y}) = (49.38394909, 22.14429463)$.
2. We found that the Cusp-Bogdanov-Takens bifurcation is persist to the perturbation for $\epsilon > 0$.

References

1. Broer, et al., *Bifurcation of a predator-prey model with non-monotonic response function*, Comptes Rendus Mathematique, V. 341, (2005) 601-604.
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THANK YOU
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