Valid and Invalid Arguments

An argument is a sequence of statements such that

- all statements but the last are called *hypotheses*
- the final statement is called the *conclusion.*
- the symbol $\therefore$ read "therefore" is usually placed just before the conclusion.

Example:

$p \wedge \sim q \rightarrow r$

$p \vee q$

$q \rightarrow p$

$\therefore\ r$

An argument is said to be **valid** if
- whenever all hypotheses are true, the conclusion must be true.

Example of a valid argument (form)

$p \wedge (q \vee r)$

$\sim q$

$\therefore\ p \wedge r$

p q r	$p \wedge (q \vee r)$	$\sim q$	$p \wedge r$
T T T	T	F	T
T T F	T	F	F
T F T	T	T	T
T F F	F	T	F
F T T	F	F	F
F T F	F	F	F
F F T	F	T	F
F F F	F	T	F

An invalid argument

$$p \rightarrow q \vee \sim r$$
$$q \rightarrow p \vee r$$
$$\therefore\ p \rightarrow r$$

	p	q	r	$p \rightarrow q \vee \sim r$	$q \rightarrow p \vee r$	$p \rightarrow r$
	T	T	T	T	T	T
	T	T	F	T	F	
	T	F	T	F	T	
Invalid row	T	F	F	T	T	F
	F	T	T	T	F	
	F	T	F	T	F	
	F	F	T	T	T	T
	F	F	F	T	T	T

Tautology

- is a statement (form) that is always true regardless of the truth values of the individual statement variables.

Examples:

- $p \vee \sim p$ (eg. the number *n* is either > 0 or ≤ 0)
- $p \wedge q \rightarrow p$
- $(p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow r)$

We need to study tautologies because any valid argument is equivalent to a tautology.
In particular, every theorem we have proved is a tautology.

Example:

$p \wedge (q \vee r)$

$\sim q$

$\therefore\ p \wedge r$

is a valid argument,

$[p \wedge (q \vee r)] \wedge [\sim q] \rightarrow [p \wedge r]$

is a tautology.

In other words, an argument

H_1

H_2

…

H_n

$\therefore$ Conclusion

is valid if and only if

$H_1 \wedge H_2 \wedge \ldots \wedge H_n \rightarrow$ conclusion

is a tautology.

Nur Insani (nurinsani@uny.ac.id)

Two most important valid argument forms

Modus Ponens : *means method of affirming*

$p \rightarrow q$

p

$\therefore q$

Example: If $n \geq 5$, then $n!$ is divisible by 10.

$n = 7$

$\therefore 7!$ is divisible by 10.

Modus Tollens : *means method of denying*

$p \rightarrow q$

$\sim q$

$\therefore \sim p$

Example: If n is odd, then n^2 is odd.

n^2 is even.

$\therefore n$ is even.

More valid forms

Conjunctive simplification: p ∧ q
ॐ p

Example: The function f is 1-to-1 and continuous.
ॐ The function f is 1-to-1.

Disjunctive addition: p
ॐ p ∨ q

Example: The function f is increasing.
ॐ The function f is increasing or differentiable.

More valid forms

Conjunctive addition: p
q
ॐ p ⮝
q

Example: *n* is an integer, *n* is positive.
ॐ *n* is a positive integer.

Disjunctive syllogism: p ⮟ q
„q
ॐ p

Example: The graph of this equation may be a circle or an ellipse.
The graph of this equation cannot be a circle.
ॐ The graph must be an (true) ellipse.

Hypothetical syllogism:

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

Proof by cases:

$p \vee q$

$p \rightarrow r$

$q \rightarrow r$

$\therefore r$

Example:

n is either odd or even.

If *n* is odd, then *n*(*n*-1) is even.

If *n* is even, then *n*(*n*-1) is even.

Therefore *n*(*n*-1) is always even.

Rule of contradiction:

$\neg p \rightarrow c$

$\therefore p$

Example:

If $\sqrt{2}$ is not irrational, then there exists whole numbers *a*, *b* that are relatively prime and are both even.

Therefore $\sqrt{2}$ must be irrational.

A valid argument with a false conclusion.

The following argument is valid by modus ponens, but since its hypothesis is false, so is its conclusion.

If p is prime, then $2^p – 1$ is also prime (False).
11 is prime (True).
Therefore $2^{11} – 1$ is prime (False).

Actually, $2^{11} – 1 = 2047 = 23 \times 89$ is not prime.

Note: Any prime of the form $2^p – 1$ is called a Mersenne prime, the largest one up to date is $2^{6972593} – 1$ (discovered on 6-1-99)