Keabsahan Argumen

Taken from “ Rules of Inference by” Rosen 1.5

Proofs in mathematics are *valid arguments*

An *argument* is a sequence of statements that end in a conclusion

By *valid* we mean the conclusion must follow from the truth of the preceding statements or premises

In other words, an argument form is *valid* means that for any substitution of statement variables,
if the premises are true,
then the conclusion is also true.

We use *rules of inference* to construct valid arguments

The rules of inference

Rule of inference	Tautology	Name
$p \to q$ p $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$-q$ $p \to q$ $\therefore -p$	$[-q \wedge (p \to q)] \to -p$	Modus tollen
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $-p$ $\therefore q$	$((p \vee q) \wedge -p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $-p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (-p \vee r)] \to (q \vee r)$	Resolution

Another view on what we are doing

You might think of this as some sort of game.

You are given some statement, and you want to see if it is a valid argument and true

You translate the statement into argument form using propositional variables, and make sure you have the premises right, and clear what is the conclusion

You then want to get from premises/hypotheses (A) to the conclusion (B) using the rules of inference.

So, get from A to B using as "moves" the rules of inference

Valid Arguments in Propositional Logic

Is this a valid argument?

If you listen you will hear what I'm saying
You are listening
Therefore, you hear what I am saying

Let p represent the statement "you listen"
Let q represent the statement "you hear what I am saying"

The argument has the form:

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad\quad} \\ \therefore q \end{array}$$

Valid Arguments in Propositional Logic

$$p \to q$$
$$p$$
$$\therefore q$$

$((p \to q) \wedge p) \to q$ is a tautology (always true)

p	q	$p \to q$	$(p \to q) \wedge p$	$((p \to q) \wedge p) \to q$
S	S	B	S	B
S	B	B	S	B
B	S	S	S	B
B	B	B	B	B

This is another way of saying that

$p \to q, p \models q$

$\therefore$ *therefore*

Valid Arguments in Propositional Logic

When we replace statements/propositions with propositional variables we have an ***argument form***.

Definition:
An argument form is a sequence of compound
All but the final proposition are called ***premises***.
The last proposition is the ***conclusion***
The derivative of a conclusion depends on a set of premises which are always true

The argument is valid if the truth of all premises implies the conclusion is true

Valid Arguments in Propositional Logic

Example:

Is this argument valid?

$a \rightarrow b \vee c, b \rightarrow -a, d \rightarrow -c, a \models -d$

Solution:

1.	$a \rightarrow b \vee c$	p
2.	$b \rightarrow -a$	p
3.	$d \rightarrow -c$	p
4.	a	p
5.	$b \vee c$	Detachment law using step 1 & 4
6.	-b	Modus tollendo tollens using step 2 & 4
7.	c	Modus tollendo ponens using step 5 & 6
8.	-d	Modus tollendo tollens using step 3 & 7

Therefore, the above argument is valid.

But we could use a truth table. Why not?

Truth Table:

The validity of an argument can be shown by showing its compound proposition as a tautology. From the example above:

$$((a \rightarrow b \vee c) \;\&\; (b \rightarrow -a) \;\&\; (d \rightarrow -c) \;\&\; a \quad -d$$

Assume that all premises are true.

Step 1	(a	→	b	∨	c)	&	(b	→	-a)	&	(d	→	-c)	&	a	→	-d
1		B						B				B				B	
2	B								S								
3				B			S										
4					B												
5													S				
6											S						
7																	B

As the premises are true and the conclusion is also true, then the compound proposition is also true.
Therefore, the compound proposition is a tautology.

Valid Arguments in Propositional Logic

Assume that compound propositions is denoted by A, C, Q, P, ...
Such as a → b ∨ c is denoted by A.

Theorem 1

- $A \models Q$ is a valid argument if and only if $A \rightarrow Q$ is a tautology.
- $A_1, A_2, \ldots A_m \models Q$ is a valid argument if and only if $A_1 \wedge A_2 \wedge \ldots \wedge A_m \rightarrow Q$ is a tautology

Theorem 2

- $A_1, A_2, \ldots, A_{m-1}, A_m \models Q$ is a valid argument if and only if $A_1 \wedge A_2 \wedge \ldots \wedge A_{m-1} \wedge A_m \rightarrow Q$ is a valid argument .

Collorary

- $A_1, A_2, \ldots, A_{m-1}, A_m \models Q$ is a valid argument if and only if $A_1, A_2, \ldots, A_{m-1} \models (A_m \rightarrow Q)$ is a valid argument .

Valid Arguments in Propositional Logic

Theorem 3

- $A_1, A_2, \dots A_m \models A_i$ for $i = 1,2,3 \dots m$ is a valid argument.
- If $A_1, A_2, \dots A_m \models Q_j$ for $j = 1,2,3 \dots p$ dan $Q_1, Q_2, \dots Q_m \models C$ then

  $A_1, A_2, \dots A_m \models C$ is a valid argument.

Example: Construct the derivation of the conclusion from the argument below!

If Cican is the 3rd winner, then if Dedi is 2nd winner then Edi will be the 4th winner. Gino will not be the 1st winner or Cica will be the 3rd winner.

The real situation: Dedi is the 2nd winner.

The conclusion: If Gino is the 1st winner then Edi will be the 4th winner.

<u>Solution</u>:

Let: c = Cica is the 3rd winner

d = Dedi is 2nd winner

e = Edi is the 4th winner

g = Gino is the 1st winner

Valid Arguments in Propositional Logic

The argument:

$$c \rightarrow (d \rightarrow e), -g \vee c, d \quad |= \quad g \rightarrow e$$

According to Theorem 2, the argument above is equivalent with

$$c \rightarrow (d \rightarrow e), -g \vee c, d, g \quad |= \quad e$$

The construction of the conclusion is then:

1.	$c \rightarrow (d \rightarrow e)$	p
2.	$-g \vee c$	p
3.	d	p
4.	g	p
5.	c	2 & 4 Modus tollendo ponens
6.	$d \rightarrow e$	1 & 5 Modus ponendo ponens
7.	e	3 & 6 Modus ponendo ponens

The construction above is also state the validity of the argument:

$$c \rightarrow (d \rightarrow e), -g \vee c, d \quad |= \quad g \rightarrow e$$

Valid Arguments in Propositional Logic

The argument:

$$c \rightarrow (d \rightarrow e), -g \vee c, d \models g \rightarrow e$$

According to Theorem 2, the argument above is equivalent with

$$c \rightarrow (d \rightarrow e), -g \vee c, d, g \models e$$

The construction of the conclusion is then:

1.	$c \rightarrow (d \rightarrow e)$	p
2.	$-g \vee c$	p
3.	d	p
4.	g	p
5.	c	2 & 4 Modus tollendo ponens
6.	$d \rightarrow e$	1 & 5 Modus ponendo ponens
7.	e	3 & 6 Modus ponendo ponens

The construction above is also state the validity of the argument:

$$c \rightarrow (d \rightarrow e), -g \vee c, d \models g \rightarrow e$$

Valid Arguments in Propositional Logic

To make it easier, let's we follow this 3 rules:

1. **p rule:** A single proposition or compound prepositions is a premise (shorted "p")

2. **t rule:** A single proposition or compound prepositions $A_1, A_2, \ldots A_m$ as premises to derive C such as $A_1, A_2, \ldots A_m \mid\rightarrow C$ is a tautology (shorted "t")

1. **cp rule:** If C is a conclusion from a set of premises $A_1, A_2, \ldots A_m$ and D then $D \rightarrow C$ is the conclusion from premises $A_1, A_2, \ldots A_m$. This rule is called the rule of conditional proof), & it is derive from Theorem 2 & its Colorary.

Valid Arguments in Propositional Logic

From the previous example, we can write as:

1.	$c \rightarrow (d \rightarrow e)$	p
2.	$-g \vee c$	p
3.	d	p
4.	g	p (additional premise)
5.	c	2, 4 t
6.	$d \rightarrow e$	1, 5 t
7.	e	3, 6 t
8.	$g \rightarrow e$	4,7 cp

Note:

- Cp rule is only used for a conclusion in the implication form.
- The use of cp rule in the step 8, the previous premise must be taken from the additional premise (step 4).

Valid Arguments in Propositional Logic

Sometimes, the conclusion of an argument is not in the implication form. Thus, if we want to use cp rule, we should change the argument in to an implication form that is equivalent.

Example:

Construct the derivation of the conclusion from the argument :

$$a \vee e,\ a \rightarrow c,\ e \rightarrow d \quad |= \quad c \vee d$$

Solution:

The above argument is equivalent with:

$$a \vee e,\ a \rightarrow c,\ e \rightarrow d \quad |= \quad -c \rightarrow d$$

The above argument is equivalent with:

$$a \vee e,\ a \rightarrow c,\ e \rightarrow d \ \ |= \ \ -c \rightarrow d$$

The construction:

1.	$a \vee e$	p
2.	$a \rightarrow c$	p
3.	$e \rightarrow d$	p
4.	-c	p (additional premise)
5.	-a	2, 4 t
6.	e	1, 5 t
7.	d	3, 6 t
8.	$-c \rightarrow d$	4,7 cp
9.	$c \vee d$	8t

However, we can use other way without using cp rule:

The previous argument :

$$a \vee e, a \rightarrow c, e \rightarrow d \quad |= \quad c \vee d$$

The construction:

1.	$a \vee e$	p
2.	$a \rightarrow c$	p
3.	$e \rightarrow d$	p
4.	$e \vee a$	1t
5.	$-e \rightarrow a$	4 t
6.	$-e \rightarrow c$	2, 5 t
7.	$-c \rightarrow e$	6 t
8.	$-c \rightarrow d$	3,7 t
9.	$c \vee d$	8t

Consistency of A set of Premises & Undirected Proof

Now, what happen if we can’t proof the validity of derivation of a conclusion from a set of premises?

What happen if a set of premises is not enough to obtain an expected conclusion?

The failure in in proving the validity of an argument could be:

- We can’t proof it
- the premises are not enough to produce an expected conclusion. This set of premises is called inconsistent.

A set of premises is called **inconsistent** if among those premises has a **false** value.

Consistency of A set of Premises & Undirected Proof

modus ponens

$$p \rightarrow q$$
$$p$$
$$\therefore q$$

$$\sqrt{2} > \frac{3}{2} \rightarrow (\sqrt{2})^2 > \left(\frac{3}{2}\right)^2$$
$$\sqrt{2} > \frac{3}{2}$$
$$\therefore 2 > \frac{9}{4}$$

A valid argument can lead to an incorrect conclusion if one of its premises is wrong/false!

$$p: \sqrt{2} > \frac{3}{2}$$

$$q: \quad 2 > \left(\frac{3}{2}\right)^2$$

$$p \rightarrow q$$

The argument is valid as it is constructed using modus ponens
But one of the premises is false (p is false)
So, we cannot derive the conclusion

- How to check if a set of premises is consistent or not?

  Assume that all the premises are true.

  If we obtain a contradiction from those premises when we do logically derivation, then those premises are inconsistent.

- The rule of how we derive a contradiction is the same with the way of how we derive a conclusion from an argument (proofing the validity of an argument).

- Check on example 5.12 (page 74).

- Other ways: using the truth table from its compound proposition form.

Check whether these premises are consistent or not?

$a \rightarrow b,\ b \rightarrow c,\ d \rightarrow -c,\ a \,\&\, d$

The construction:

1.	$a \rightarrow b$	p
2.	$b \rightarrow c$	p
3.	$d \rightarrow -c$	p
4.	a & d	p
5.	a	4 t
6.	$a \rightarrow b$	1,2 t
7.	c	5,6 t
8.	d	4 t
9.	-c	3,8 t
10.	c & -c	7,9 t

Thus, the premises are incosistent.

- How about if we check it using truth table?
  - Find its compound proposition form
  - Assume each premise is true.
  - Form the truth table & check very each single proposition → you'll see the contradiction (if any).

Truth Table: assume that all premises are B.

a → b, b → c, d → -c, a & d

Step	(a	→	b)	&	(b	→	c)	&	(d	→	-	c)	&	(a	&	d)
1		B				B				B					B	
2														B		B
3	B								B							
4			B								B					
5					B							S				
6							B									

At step 5 & 6, the value of c is both B & S, it's a contradiction.
Therefore, the assumption must be refused, thus, not alll premises are true.

Consistency of A set of Premises & Undirected Proof

Theorem 5.4

A set of $A_1, A_2, \ldots A_m$ is inconsistent if from this set we can derive a contradiction.

Example:

Show that these premises are inconsistent:

$$a \rightarrow b,\ b \rightarrow c,\ -c \vee d,\ -a \rightarrow d,\ -d$$

The solution:

1.	$a \rightarrow b$	p
2.	$b \rightarrow c$	p
3.	$-c \vee d$	p
4.	$-a \rightarrow d$	p
5.	$-d$	p
6.	$-(-a)$	
7.	a	
8.	b	
9.	c	
10.	d	
11.	$-d$ & d	

Undirected Proof

- Now, we are going to use the rule of conditional proof (cp) & the definition of an inconsistent set of premises to bring us to an important proofing method in Maths:

  undirected proof /

  proof with contradiction /

  reductio ad absurdum

Undirected Proof

Example:

Proof the validity of the argument below using reductio ad absurdum:

$$a \rightarrow b \vee c,\ b \rightarrow -a,\ d \rightarrow -c \quad |= \quad a \rightarrow -d$$

Remember: $-(a \rightarrow -d)$ ek a & d. We must show that the premises are inconsistent.

The solution:

1.	$a \rightarrow b \vee c$	p
2.	$b \rightarrow -a$	p
3.	$d \rightarrow -c$	p
4.	a & d	p (add p)
5.	a	p
6.	$b \vee c$	1,5 t
7.	d	4 t
8.	-c	3,7 t

Undirected Proof

The solution:

1.	$a \rightarrow b \vee c$	p
2.	$b \rightarrow$ -a	p
3.	$d \rightarrow$ -c	p
4.	a & d	p (add p)
5.	a	p
6.	$b \vee c$	1,5 t
7.	d	4 t
8.	-c	3,7 t
9.	b	6,8 t
10.	-a	5,9 t
11.	a & -a	5, 10 t
12.	a & d $\rightarrow$ a & -a	4, 11 t
13.	- (a & d)	12 t
14.	$a \rightarrow$ -d	13 t

Hint: take the contradiction/pengandaian harus diingkar

- **Steps to proof with reductio ad absurdum method:**
  - **Take the negation from the conclusion, & use it as an additional premise;**
  - **Derive a contradiction;**
  - **If the contradiction is achieved, then the neglect conclusion is the logic conclusion from the specified premises.**

Undirected Proof

Theorem 5.4

$A_1, A_2, \ldots A_m \models C$ is a valid argument, if we can derive a contradiction from $A_1, A_2, \ldots A_m \;\&\; -C$