Contoh Persoalan Integral Lipat dua dan tiga

Selesaikanlah soal-soal berikut ini:

1. $\iint dx\, dy$

Penyelesaian:

$$\iint dx\, dy = \int (x + c_1)\, dy$$

$$= (x + c_1)y + c_2 = xy + yc_1 + c_2$$

2. $\iint \rho \cos\theta \; d\rho\, d\theta$

Penyelesaian:

$$\iint \rho \cos\theta \; d\rho\, d\theta = \int (\tfrac{1}{2}\rho^2 \cos\theta + c_1)\, d\theta$$

$$= \tfrac{1}{2}\rho^2 \sin\theta + \theta\, c_1 + c_2$$

3. $\iint e^{3t-7} dS\, dt$

Penyelesaian:

$$\iint e^{3t-7} dS\, dt = \int (Se^{3t-7} + c_1)\, dt = \boxed{\int Se^{3t-7} dt} + \int c_1\, dt$$

menggunakan cara pemisalan

Misal: u = 3t–7 jika diderivatifkan dan dengan operasi aljabar maka akan didapat $dt = \dfrac{du}{3}$ sehingga integralnya menjadi $\int Se^u \frac{du}{3} + \int c_1 dt$ jika diintegralkan diperoleh:

$\frac{1}{3} S\, e^u + t\, c_1 + c_2$ ingat u = 3t–7

Jadi hasil akhir dari soal ini adalah $\frac{1}{3} S\, e^{3t-7} + t\, c_1 + c_2$

4. $\iint \rho \arctan\phi \, d\phi \, d\rho$

Penyelesaian:

$$\iint \rho \arctan\phi \, d\phi \, d\rho = \int \rho \boxed{\int \arctan\phi \, d\phi} \, d\rho$$

menggunakan integral parsial

Integral yang dikotak merah menggunakan integral parsial, partisinya adalah diambil: $u = \arctan\phi$ dan $dv = d\phi$, jika u diderivatifkan diperoleh: $du = \dfrac{d\phi}{1+\phi^2}$ dan jika dv diintegralkan $\int dv = \int d\phi$ maka didapat $v = \phi$ sehingga integral yang ada dalam kotak merah menjadi

$$\rho \int \arctan\phi \, d\phi = \rho\left(\arctan\phi \,.\, \phi - \boxed{\int \phi \frac{1}{1+\phi^2} d\phi}\right)$$

menggunakan cara pemisalan

misal: $u = 1+\phi^2$ jika diderivatifkan dan dengan operasi aljabar maka akan didapat $d\phi = \dfrac{du}{2\phi}$ sehingga integralnya menjadi

$$\int \frac{\phi}{u}\frac{du}{2\phi} = \int \frac{1}{2}\frac{du}{u} = \tfrac{1}{2}\ln u \quad \text{ingat } u = 1+\phi^2$$

maka hasilnya:

$$\rho(\phi \arctan\phi - \tfrac{1}{2}\ln(1+\phi^2)) + c_1$$

Hasil tersebut diintegralkan kembali terhadap ρ,

$$\int \rho(\phi \arctan\phi - \tfrac{1}{2}\ln(1+\phi^2) + c_1)\, d\rho = \frac{1}{2}\rho^2(\phi \arctan\phi - \tfrac{1}{2}\ln(1+\phi^2) + \rho c_1 + c_2$$

Jadi hasil akhir dari soal ini adalah $\frac{1}{2}\rho^2(\phi \arctan\phi - \tfrac{1}{2}\ln(1+\phi^2) + \rho c_1 + c_2$

5. $\displaystyle\int_{-3}^{6}\int_{4-3y}^{y^2+3} dx\, dy$

Penyelesaian:

$$\int_{-3}^{6}\int_{4-3y}^{y^2+3} dx\, dy = \int_{-3}^{6} x\Big|_{4-3y}^{y^2+3} dy = \int_{-3}^{6}((y^2+3)-(4-3y))\, dy = \int_{-3}^{6}(y^2+3y-1)\, dy$$

$$= \tfrac{1}{3}y^3 + \tfrac{3}{2}y^2 - y\Big|_{-3}^{6} = 120 - 7{,}5 = 112{,}5$$

6. $\int_0^{\frac{\pi}{2}}\int_0^{\sin\theta} \rho\cos\theta\ d\rho\, d\theta$

Penyelesaian:

$$\int_0^{\frac{\pi}{2}}\int_0^{\sin\theta} \rho\cos\theta\ d\rho\, d\theta = \int_0^{\frac{\pi}{2}} \left. \tfrac{1}{2}\rho^2\cos\theta \right|_0^{\sin\theta} d\theta = \boxed{\int_0^{\frac{\pi}{2}} \tfrac{1}{2}\sin^2\theta\cos\theta\ d\theta}$$

menggunakan cara pemisalan

misal: $u = \sin\theta$ jika diderivatifkan dan dengan operasi aljabar maka akan didapat $d\theta = \dfrac{du}{\cos\theta}$ sehingga integralnya menjadi $\int \tfrac{1}{2}u^2.\cos\theta.\dfrac{du}{\cos\theta}$

sehingga diperoleh: $\int \tfrac{1}{2}u^2 du = \tfrac{1}{6}u^3$ ingat: $u = \sin\theta$

maka hasilnya:

$$\left. \tfrac{1}{6}\sin^3\theta \right|_0^{\frac{\pi}{2}} = \tfrac{1}{6}(\sin^3\tfrac{\pi}{2} - \sin^3 0) = \tfrac{1}{6}(1-0)$$

$$= \frac{1}{6}$$

7. $\int_0^{1}\int_0^{e^{2t+7}} e^{3t-7} dS\, dt$

Penyelesaian:

$$\int_0^{1}\int_0^{e^{2t+7}} e^{3t-7} dS\, dt = \int_0^{1} \left. S\, e^{3t-7} \right|_0^{e^{2t+7}} dt = \int_0^{1} (e^{2t+7}.e^{3t-7})\, dt = \boxed{\int_0^{1} e^{5t}\, dt}$$

menggunakan cara pemisalan

Dengan pemisalan $u = 5t$ sehingga diperoleh: $\left. \tfrac{1}{5}e^{5t} \right|_0^1 = \tfrac{1}{5}(e^5 - 1)$

maka hasilnya: $\dfrac{1}{5}(e^5 - 1)$

8. $\displaystyle\int_0^{\frac{\pi}{2}}\int_2^{4\cos\phi}\rho^3 d\rho\, d\phi$

Penyelesaian:

$$\int_0^{\frac{\pi}{2}}\int_2^{4\cos\phi}\rho^3 d\rho\, d\phi = \int_0^{\frac{\pi}{2}}\left[\tfrac{1}{4}\rho^4\right]_2^{4\cos\phi} d\phi = \int_0^{\frac{\pi}{2}}(64\cos^4\phi - 4)\, d\phi = \boxed{\int_0^{\frac{\pi}{2}} 64\cos^4\phi\, d\phi} - \int_0^{\frac{\pi}{2}} 4\, d\phi$$

menggunakan sifat-sifat trigonometri

ingat salah satu sifat trigonometri, yaitu: $\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$

sehingga $\cos^4\alpha = (\cos^2\alpha)^2 = \left(\frac{1}{2}(1+\cos 2\alpha)\right)^2$

$$= \tfrac{1}{4}(1 + 2\cos 2\alpha + \cos^2 2\alpha)$$

$$= \tfrac{1}{4}\left(1 + 2\cos 2\alpha + \tfrac{1}{2}(1 + \cos 4\alpha)\right)$$

integralnya menjadi:

$$\int_0^{\frac{\pi}{2}} 16\left(1 + 2\cos 2\phi + \tfrac{1}{2}(1+\cos 4\phi)\right)d\phi - \int_0^{\frac{\pi}{2}} 4\, d\phi = \int_0^{\frac{\pi}{2}} 16\left(1 + 2\cos 2\phi + \tfrac{1}{2}(1+\cos 4\phi)\right)d\phi - 4\phi\Big|_0^{\frac{\pi}{2}}$$

$$= \int_0^{\frac{\pi}{2}}\left(24 + \boxed{32\cos 2\phi} + \boxed{8\cos 4\phi}\right)d\phi - 4\phi\Big|_0^{\frac{\pi}{2}}$$

menggunakan cara pemisalan, diambil: u = 2ϕ

menggunakan cara pemisalan, diambil: u = 4ϕ

Sehingga hasil integralnya adalah

$$= \left(24\phi + 16\sin 2\phi + 2\sin 4\phi\right)\Big|_0^{\frac{\pi}{2}} - 4\phi\Big|_0^{\frac{\pi}{2}}$$

$$= (12\pi + 0 + 0) - (0 + 0 + 0) - 2\pi$$

$$= 10\pi$$

9. $\iiint dx\, dz\, dy$

Penyelesaian:

$$\iiint dx\, dz\, dy = \iint (x + c_1)\, dz\, dy = \int \left((x + c_1)z + c_2\right)dy$$

$$= (x + c_1)\, zy + yc_2 + c_3$$

$$= xyz + zyc_1 + yc_2 + c_3$$

10. $\iiint (xy + yz + xz)\, dy\, dz\, dx$

Penyelesaian:

$$\iiint (xy + yz + xz)\, dy\, dz\, dx = \iint (\tfrac{1}{2} xy^2 + \tfrac{1}{2} y^2 z + xzy + c_1)\, dz\, dx$$
$$= \int (\tfrac{1}{2} xy^2 z + \tfrac{1}{4} y^2 z^2 + \tfrac{1}{2} xz^2 y + zc_1 + c_2) dx$$
$$= \tfrac{1}{4} x^2 y^2 z + \tfrac{1}{4} xy^2 z^2 + \tfrac{1}{4} x^2 z^2 y + xzc_1 + xc_2 + c_3$$

11. $\int_0^{\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec\phi} \sin 2\phi\, d\rho\, d\phi\, d\theta$

Penyelesaian:

$$\int_0^{\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec\phi} \sin 2\phi\, d\rho\, d\phi\, d\theta = \int_0^{\pi} \int_0^{\frac{\pi}{4}} \left(\rho\Big|_0^{\sec\phi} \sin 2\phi \right) d\phi\, d\theta$$
$$= \int_0^{\pi} \int_0^{\frac{\pi}{4}} \left(\sec\phi \,.\, \sin 2\phi \right) d\phi\, d\theta = 2\int_0^{\pi} \int_0^{\frac{\pi}{4}} \sin\phi\, d\phi\, d\theta$$
$$= 2\int_0^{\pi} \left(-\cos\phi\, \Big|_0^{\frac{\pi}{4}} \right) d\theta = 2\int_0^{\pi} \left(-\tfrac{1}{2}\sqrt{2} \right) d\theta$$

$$= 2\left(-\tfrac{1}{2}\sqrt{2} \right)\theta\Big|_0^{\pi} = (2-\sqrt{2})\pi$$

12. $\int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2-x^2}} \int_{2x^2+y^2}^{4-y^2} dz\, dy\, dx$

Penyelesaian:

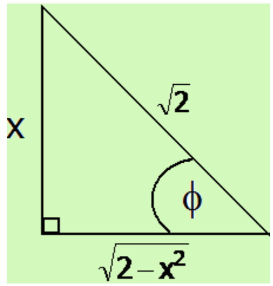
$$\int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2-x^2}} \int_{2x^2+y^2}^{4-y^2} dz\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2-x^2}} (4 - y^2) - (2x^2 + y^2)\, dy\, dx$$
$$= \int_0^{\frac{\pi}{2}} \left((4y - 2x^2 y - \tfrac{2}{3} y^3)\Big|_0^{\sqrt{2-x^2}} \right) dx = \boxed{\int_0^{\frac{\pi}{2}} \sqrt{(2-x^2)^3}\, dx}$$

menggunakan substitusi trigonometri, diambil: x = $\sqrt{2}$ sin ϕ

dengan mengambil $x = \sqrt{2}\,\sin\phi$, jika diderivatifkan diperoleh $dx = \sqrt{2}\cos\phi\, d\phi$

disamping itu didapat juga: $\sin\phi = \dfrac{x}{\sqrt{2}}$

sehingga dapat dibuat segitiga siku-siku sebagaimana gambar disamping, dari segitiga tersebut diperoleh:



$$\cos\phi = \frac{\sqrt{2-x^2}}{\sqrt{2}} \Leftrightarrow \sqrt{2-x^2} = \sqrt{2}\cos\phi$$

$$\tan\phi = \frac{x}{\sqrt{2-x^2}}$$

sehingga integralnya menjadi:

$$= \int_0^{\frac{\pi}{2}} \left(\sqrt{2}\cos\phi\right)^3 \sqrt{2}\cos\phi\, d\phi = \boxed{\int_0^{\frac{\pi}{2}} 4\cos^4\phi\, d\phi}$$

Dengan langkah seperti pada contoh no.8 maka diperoleh bentuk integral berikut ini:

Penyelesaiannya seperti soal no.8

$$= \int_0^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\cos 2\phi + \tfrac{1}{2}\cos 4\phi\right) d\phi$$

$$= \left(\frac{3}{2}\phi + \sin 2\phi + \tfrac{1}{8}\sin 4\phi\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{3}{4}\pi$$