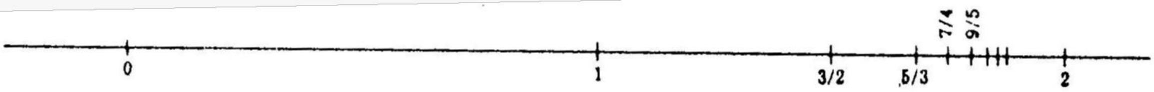
1. LIMITS

LIMIT OF A SEQUENCE. As consecutive points, given by the terms of the sequence

$$1, 3/2, 5/3, 7/4, 9/5, \ldots, 2-1/n \qquad (1)$$

are located on a number scale, it is noted that they cluster about the point 2 in such a way that there are points of the sequence whose different from 2 is less than any preassigned positive number, however small.



For example, the point 2001/1001 and all subsequent are at distance < 1/1000 from 2, the point 20 000 001/10 000 001 and all subsequent points are at a distance < 1/10 000 000 from 2, and so on. This state of affairs is indicated by saying that *the limit of the sequence is 2.*

If x is variable whose range is the sequence (*1*), we say that *x approaches 2 as limit or x tends to 2 as limit* and write $x \to 2$.

The sequence (*1*) does not contain its limit 2 as a term. On the other hand, the sequence 1, 1/2, 1, ¾, 1, 5/6, 1, … has 1 limit and every odd number term is 1. Thus, a sequence may or may not reach its limit. Hereinafter, the statement $x \to a$ will be understood to imply $x \neq a$, that is, *it is to be understood that any given arbitrary sequence does not contain its limit as a term.*

LIMIT OF A FUNCTION. Let $x \to 2$ over the sequence (1); then $f(x) = x^2 \to 4$ over the sequence 1, 9/4, 25/9, 49/16, …, $(2-1/n)^2$, … Now let $x \to 2$ over the sequence

$$2.1, 2.01, 2.001, 2.0001, \ldots, 2 + 1/10^n, \ldots \qquad (2)$$

then $x^2 \to 4$ over the sequence 4.41, 4.0401, 4.004001, …, $(2 + 1/10^n)^2$, … It would seem reasonable to expect that would approach 4 as limit however x may approach 2 as limit. Under this assumption, we say "the limit, as x approaches 2, of x^2 is 4" and write

$$\lim_{x \to 2} x^2 = 4$$

RIGHT AND LEFT LIMITS. As $x \to 2$ over the sequence (1), its value is always less than 2. We say that greater than 2. We say that *x approaches 2 from the right* and write $x \to 2^+$. Clearly, the statement $\lim_{x \to a} f(x)$ exists implies that both the *left limit* $\lim_{x \to a^-} f(x)$ and the $\lim_{x \to a^+} f(x)$ exist and are equal. However, the existence of the right (left) limit does not imply the existence of the left (right) limit.

Example 1:

The function $f(x)=\sqrt{9-x^2}$ has the interval $-3 \le x \le 3$ as domain of definition. If a is any number on the open interval $-3 \le x \le 3$, then $\lim\limits_{x\to 3}\sqrt{9-x^2}$ exists and is equal to $\sqrt{9-a^2}$. Now consider a 3. First, let x approach 3 from the left; then $\lim\limits_{x\to 3^-}\sqrt{9-x^2}=0$. Next, let x approach 3 from the right; then $\lim\limits_{x\to 3^+}\sqrt{9-x^2}$ does not exist since for $x > 3$, $\sqrt{9-x^2}$ is imaginary. Thus, $\lim\limits_{x\to 3}\sqrt{9-x^2}$ does not exist.

Similarly, $\lim\limits_{x\to -3^+}\sqrt{9-x^2}$ exists and is equal to 0 but $\lim\limits_{x\to -3^-}\sqrt{9-x^2}$ and thus $\lim\limits_{x\to -3}\sqrt{9-x^2}$ do not exist.

THEOREMS ON LIMITS. The following theorems on limits are listed for further reference.

1. If $f(x)=c$, a constant, then $\lim\limits_{x\to a} f(x)=c$
2. If $\lim\limits_{x\to a} f(x)=A$ and $\lim\limits_{x\to a} g(x)=B$, then:
3. $\lim\limits_{x\to a} k.f(x)=kA$, k being any constant.
4. $\lim\limits_{x\to a}\left|f(x)\pm g(x)\right|=\lim\limits_{x\to a} f(x)\pm\lim\limits_{x\to a} g(x)=A\pm B$.
5. $\lim\limits_{x\to a}\left|f(x).g(x)\right|=\lim\limits_{x\to a} f(x).\lim\limits_{x\to a} g(x)=A.B$.
6. $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}=\dfrac{A}{B}$, provided $B\neq 0$.
7. $\lim\limits_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim\limits_{x\to a} f(x)}=\sqrt[n]{A}$, provided $\sqrt[n]{A}$ is a real number.

SOLVED PROBLEMS

1. Determine the limit of each of the following sequences:

(a) 1, 1/2, 1/3, 1/4, 1/5, …

(b) 1, 1/4, 1/9, 1/16, 1/25, …

(c) 2, 5/2, 8/3, 11/4, 14/5, …

(d). 5, 4, 11/3, 7/2, 17/5, …

(e). 1/2, 1/4, 1/8, 1/16, 1/32, …

(f). .9, .99, .999, .9999, .999 99, …

(a) The general term is 1/n. As n rakes on the values 1, 2, 3, 4, … in turn, 1/n decrease but remains positive. The limit is 0.

(b) The general term is $(1/n)^2$; the limit is 0.

(c) The general term is 3 – 1/n; the limit is 3.

(d) The general term is 3 + 2/n; the limit is 3.

(e) The general term is $1/2^n$; as in *(a)* the limit is 0.

(f) The general term is $1 - 1/10^n$; the limit is 1.

2. Describe the behaviour of $y = x + 2$ as x ranges over values of each of the sequences of Prob. 1.

   (a) $y \to 2$ over the sequence 3, 5/2, 7/3, 9/4, 11/5, …, 2 + 1/n, …

   (b) $y \to 2$ over the sequence 3, 9/4, 19/9, 33/16, 51/25, …, $2 + 1/n^2$, …

   (c) $y \to 5$ over the sequence 4, 9/2, 14/3, 19/4, 24/5, …5 - 1/n, …

   (d) $y \to 5$ over the sequence 7, 6, 17/3, 11/2, 27/5, …, 5 + 2/n, …

   (e) $y \to 2$ over the sequence 5/2, 9/4, 17/8, 33/16, 65/32, …, $2 + 1/2^n$, …

   (f) $y \to 3$ over the sequence 2.9, 2.99, 2.999, 2.9999, …, $3 - \frac{1}{10^n}$, …

3. Evaluate

   (a) $\lim_{x\to 2} 5x = 5\lim_{x\to 2} x = 5.2 = 10$

   (b) $\lim_{x\to 2}(2x+3) = 2\lim_{x\to 2} x + \lim_{x\to 2} 3$

   $= 2.2 + 3 = 7$

   (c) $\lim_{x\to 2}(x^2 - 4x + 1) = 4 - 8 + 1 = -3$

   (d) $\lim_{x\to 3} \frac{x-2}{x+2} = \frac{\lim_{x\to 3}(x-2)}{\lim_{x\to 3}(x+2)} = \frac{1}{5}$

   (e) $\lim_{x\to -2} \frac{x^2-4}{x^2+4} = \frac{4-4}{4+4} = 0$

   (f) $\lim_{x\to 4}\sqrt{25-x^2} = \sqrt{\lim_{x\to 4}(25-x^2)} = \sqrt{9} = 3$

Note. Do not assume from these problems that $\lim_{x\to a} f(x)$ is invariably *f(a)*. By *f(a)* is meant the value of *f(x) when x = a; x is never equal to a as x* $\to$ *a.*

2. CONTINUITY

A FUNCTION $f(x)$ is said to be *continuous* at $x = x_0$, if

(i) $f(x_0)$ *is* defined, (ii) $\lim_{x \to x_0} f(x)$ exists, (iii) $\lim_{x \to x_0} f(x) = f(x_0)$

For example $f(x) = x^2 + 1$ is continuous at $x = 2$ since $\lim_{x \to x_0} f(x) = 5 = f(2)$. The condition (i) implies that a function can be continuous only at points on its domain of definition. Thus, $f(x) = \sqrt{4 - x^2}$ is not continuous at $x = 3$ since $f(3)$ is imaginary, i.e. is not defined.

A function which is continuous at every point of an interval (open or closed) is said to be continuous on that interval. A function $f(x)$ is called *continuous* if it is continuous at every point on its domain of definition. Thus, $f(x)= x^2 + 1$ and all other polynomials in x are continuous function; other examples are e^x¸ sin x, cos x.

If the domain of definition of a function is a closed interval $a \le x \le b$, condition (ii) fails at the endpoints a and b. We shall call such a function continuous if it is continuous on the open interval $a < x < b$, if $\lim_{x \to a^+} f(x) = f(a)$, and if $\lim_{x \to b^-} f(x) = f(b)$. Thus $f(x) = \sqrt{9 - x^2}$ will be called a continuous function (see example 1, chapter 2). The functions of elementary calculus are continuous on their domains of definition with the possible exception of a number of isolated points.

A FUNCTION $f(x)$ is said to be *discontinuous* at $x = x_0$ if one or more of the conditions for continuity fail there. The several types of discontinuity will be illustrated by examples:

(a) $f(x) = \dfrac{1}{x-2}$ is discontinuous at $x = 2$ since

(i) $f(2)$ is not defined (has zero as denominator)

(ii) $\lim_{x \to 2} f(x)$ does not exist (equals x).

The function is discontinuous everywhere except at $x = 2$ where it is said to have an *infinite discontinuity*. See Fig. 3-1.

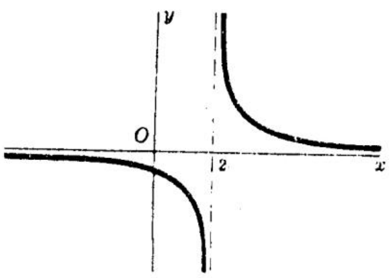


Fig. 3-1

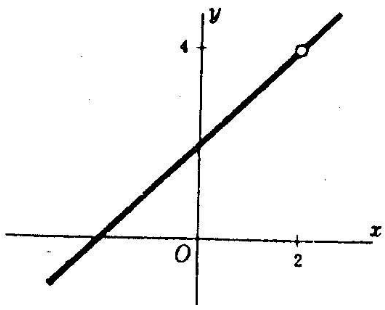


Fig. 3-2

(b) $f(x) = \dfrac{x^2 - 4}{x - 2}$ is discontinuous at *x = 2* since

(i) *f(2)* is not defined (both numerator and denominator are zero).

(ii) $\lim_{x\to 2} f(x) = 4$

The continuity here is called *removable* since it may be removed by redefining the function as $f(x)\dfrac{x^2 - 4}{x - 2}$, *x ≠ 2; f(2) = 4.* (Note that the discontinuity in *(a)* connot be so removed since the limit also does not exist.) The graphs of $f(x)\dfrac{x^2 - 4}{x - 2}$ and *g(x) = x + 2* are identical except at *x = 2* where the former has a 'hole'. Removing the discontinuity consists simply of properly filling the 'hole.'

(c) $f(x)\dfrac{x^2 - 27}{x - 3}$, *x ≠ 3; f(3) = 9* is discontinuous at *x = 3* since

(i) *f(3) = 9,* (ii) $\lim_{x\to 3} f(x) = 27$ (iii) $\lim_{x\to 3} f(x) \neq f(3)$

The discontinuity may be removed by redefining the function as $f(x)\dfrac{x^2 - 27}{x - 3}$, *x ≠ 3; f(3) = 27.*

(d) The function of Problem 9, Chapter 1, is defined for all *x > 0* but has discontinuities at *x = 1, 2, 3, …* arising form the fact that

$$\lim_{x\to s^-} f(x) \neq \lim_{x\to s^+} f(x) \quad (s \text{ any positive integer})$$

These are called *jump discontinuities*

3. THE DERIVATIVE

INCREMENTS. The *increment* Δx of a variable x is the change in x as it increase from one value $x = x_0$ to another value $x = x_1$ in its range. Here, $\Delta x = x_1 - x_0$ and we may write $x_1 = x_0 + \Delta x$.

If the variable x is given an increment Δx from $x = x_0$ (that is, if x changes from $x = x_0$ to $x = x_0 + \Delta x$) and a function a function $y = f(x)$ is thereby given an increment $\Delta y = f(x_0 + \Delta x) - f(x_0)$ from $y = f(x_0)$, the quotient

$$\frac{\Delta y}{\Delta x} = \frac{change\ in\ y}{change\ in\ x}$$

is called the *average rate of change* of the function on the interval between $x = x_0$ and $x = x_0 + \Delta x$.

Example 1:
When x is given the increment $\Delta x = 0.5$ from $x_0 = 1$, the function $y = x^2 + 2x$ is given the increment $\Delta y = f(1 + 0.5)^2 - f(1) = 5.25\text{-}3 = 2.25$. Thus, the average rate of change of y on the interval between $x = 1$ and $x = 1.5$ is $\frac{\Delta y}{\Delta x} = \frac{2.25}{0.5} = 4.5$

THE DERIVATIVE of function $y = f(x)$ with respect to x at the point $x = x_0$ is defined as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Provided the limit exists. This limit is also called the *instantaneous rate of change* (or simply, the *rate of change*) of y with respect to x at $x = x_0$.

Example 2:
Find the derivate of $y = f(x) = x^2 + 3x$ with respect to x at $x = x_0$. Use this to find the value of derivative at (a) $x_0 = 2$ and (b) $x_0 = -4$.

$$y_0 = f(x_0) = x_0^2 + 3x_0$$

$$y_0 + \Delta y = f(x_0 + \Delta x) = (x_0 + \Delta x)^2 + 3(x_0 + \Delta x)$$

$$= x_0^2 + 2x_0\Delta x + (\Delta x)^2 + 3x_0 + 3\Delta x$$

$$\Delta y = f(x_0 + \Delta x) - f(x_0) = 2x_0\Delta x + 3\Delta x + (\Delta x)^2$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = 2x_0 + 3 + \Delta x$$

The derivative at $x = x_0$ is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x_0 + 3 + \Delta x) = 2x_0 + 3$$

(a) At $x_0 = 2$, the value of the derivative is 2 . 2 + 3 = 7.

(b) At $x_0 = -4$, the value of the derivative is 2(-4) + 3 = -5.

IN FINDING DERIVATIVES it is customary to drop the subscribe 0 to obtain the derivative of $y = f(x)$ *with respect to* x as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

See note following problems 5 (*c*), Chapter 2.

The derivative of $y = f(x)$ with respect to x may be indicated by any one of the symbols

$$\frac{d}{dx} y, \frac{dy}{dx}, D_x y, y', f'(x) \text{ or } \frac{d}{dx} f(x)$$

4. DIFFERENTIATION OF ALGEBRAIC FUNCTION

A FUNCTION is said to be *differentiable at* $x = x_0$ if it has a derivative there. A function is said to be *differentiable on an interval* if it is differentiable at every point of the interval.

The functions of elementary calculus are differentiable, except possibly at certain isolated points, on their intervals of definition.

DIFFERENTIATION FORMULAS. In this formulas *u*, *v* and *w* are differentiable function of *x*.

1. $\frac{d}{dx}(c) = 0,$
2. $\frac{d}{dx}(x) = 1$
3. $\frac{d}{dx}(u + v + \ldots) = \frac{d}{dx}(u) + \frac{d}{dx}(v) + \ldots$
4. $\frac{d}{dx}(cu) = c\frac{d}{dx}(u)$
5. $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$
6. $\frac{d}{dx(uvw)} = uv\frac{d}{dx}(w) + uw\frac{d}{dx}(v) + vw\frac{d}{dx}(u)$
7. $\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{1}{c} \cdot \frac{d}{dx}(u), c \neq 0$
8. $\frac{d}{dx}\left(\frac{c}{u}\right) = c\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{c}{u^2} \cdot \frac{d}{dx}(u)$
9. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}, v \neq 0$
10. $\frac{d}{dx}(x^m) = mx^{m-1}$
11. $\frac{d}{dx}(u^m) = mu^{m-1}\frac{d}{dx}(u)$
12. $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$
13. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

INVERSE FUNCTIONS. Let $y = f(x)$ be differentiable on the interval $a \le x \le b$ and suppose that *dy/dx* does not change sign on the interval. Then from Fig. 5-1a and 5-1 b the function assumes once and only once every value between $f(a) = c$ and $f(b) = d$. Thus, for each value of *y* on the respective interval, there corresponds one and only one value of *x* and *x* is a function of *y*, say $x = g(y)$. The function $y = f(x)$ and $x = g(y)$ are called *inverse functions.*

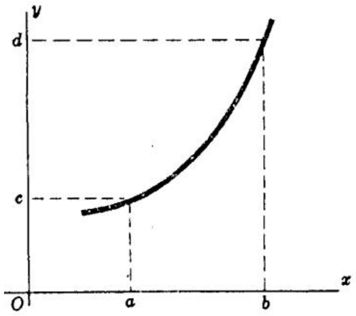


Fig. 5-1a

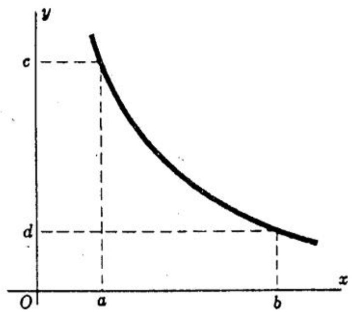


Fig. 5-1b

Example I:

(a) $y = f(x) = 3x + 2$ and $x = g(y) = f(y-2)$ are inverse function

(b) When $x \le 2$ and $y \ge -1$, $y = x^2 - 4x + 3$ and $x = 2 - \sqrt{y+1}$ are inverse function.

When $x \ge 2$ and $y \ge -1$, $y = x^2 - 4x + 3$ and $x = 2 + \sqrt{y+1}$ are inverse function.

To find *dy/dx*, given $x = g(y)$

(a) Solve for *y*, when possible, and differentiate with respect to *x;* or

(b) Differentiate $x = g(y)$ with respect to *y* and use

Example 2:

Find *dy/dx*, given $x = \sqrt{y} + 5$

Using (a) : $y = (x - 5)^2$ and $dy/dx = 2(x - 5)$

Using (b): $\frac{dx}{dy} = \frac{1}{2} y^{-1/2} = \frac{1}{2\sqrt{y}}$; then $\frac{dy}{dx} = 2\sqrt{y} = 2(x-5)$.

DIFFERENTIATION OF A FUNCTION OF A FUNCTION. If $y = f(u)$ and $u = g(x)$, then $y = f\{g(x)\}$ is a function of *x*. If *y* is a differentiable function of *u* and if *u* is a differentiable function of *x*, then $y = f\{g(x)\}$ is differentiable function of *x*_and the derivative *dy/dx* may be obtained by one of the following procedures:

(a) Express *y* explicitly in terms of *x* and differentiate.

Example 3:

If $y = u^2 + 3$ and $u = 2x + 1$, then $y = (2x + 1)^2 + 3$ and $dy/dx = 8x + 4$.

(b) Differentiate each function with respect to the independent variable and use the formula (*the chain rule*).

Example 4:

If $y = u^2 + 3$ and $u = 2x + 1$, then $\frac{dy}{du} = 2u$, $\frac{du}{dx} = 2$ and $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u = 8x + 4$

HIGHER DERIVATIVES. Let $y = f(x)$ be a differentiable function of x and let its derivative be called the *first derivative* of the function. If the first derivative is differentiable, its derivative is called the *second derivative* of the (original) function and is denoted by one of the symbols $\frac{d^2y}{dx^2}$, $y``$, or $f``(x)$. In turn, the derivative of the second derivative is called the *third derivative* of the function and is denoted by one of the symbols $\frac{d3y}{dx^3}, y''', or\ f'''(x); \ldots$

Note. The derivative of a given order at a point can exist only when the function and all derivatives of lower order are differentiable at the pont.

5. MAXIMUM AND MINIMUM VALUE

INCREASING AND DECREASING FUNCTION. A function $f(x)$ is said to be *increasing* at $x = x_0$ if for h, positive and sufficiently small, $f(x_0 - h) < f(x_0) < f(x_0+h)$. A function $f(x)$ is said to be decreasing at $x = x_0$ if for h, positive and sufficiently small, $f(x_0 - h) > f(x_0) > f(x_0+h)$

If $f`(x_0) > 0$, then $f(x)$ is an increasing function at $x = x_0$; if $f`(x_0) < 0$, then $f(x)$ is decreasing function at $x = x_0$. If $f`(x_0) = 0$, then $f(x)$ is said to be *stationary* at $x = x_0$.

A non-constant function is said to be an increasing (decreasing) function over an interval if it is increasing (decreasing or stationary at every point of the interval.

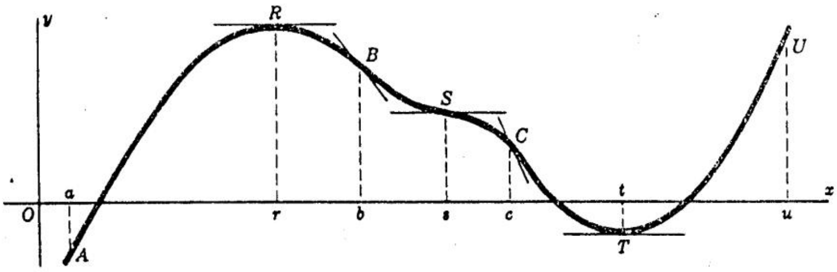


Fig. 8-1

In Fig. 8-1, the curve $y = f(x)$ is rising (the function is increasing) on the intervals $a < x < r$ and $t < x < u$; the curve is falling (the function is decreasing) on the interval $r < x < t$. The function is stationary at $x = r$, $x = s$, and $x = t$; the curve has a horizontal tangent at the points R, S, and T. The values of x, (r, s, and t), for which the function $f(x)$ is stationary ($f`(x)=0$) are more frequently called *critical values* for the function and the corresponding points (R, S, and T) of the graph are called *critical points* of the curve.

RELATIVE MAXIMUM AND MINIMUM VALUES OF A FUNCTION. A function $y = f(x)$ is said to have a *relative maximum (relative minimum)* value at $x= x_0$ if $f(x_0)$ is greater (smaller) than immediately proceeding *and* succeeding values of the function.

In Fig. 8-1, $R(r, f(r))$ is relative maximum point of the curve since $f(r) > f(x)$ on any sufficiently small neighbourhood $0 < |x - r| < 8$. We shall say that $y = f(x)$ has a *relative maximum value* ($= f(r)$)when $x = r$. In the same figure, $T (t, f(t))$ is a relative minimum point of the curve since $f(t) < f(x)$ on any sufficiently small neighbourhood $0 < |x - r| < 8$. We shall say that $y = f(x)$ has a *relative minimum value* ($=f(t)$) when $x = t$. Note that R joins an arch AR which is rising ($f`(x)>0$), and an arch $RB$ which is falling ($f`(x)<0$) while T joins an arch CT which is falling [$f`(x) < 0$] and an arch $TU$ which is rising [$f`(x) > 0$]. At S two arcs BS and SC both of which are falling are joined; S is neither a relative maximum nor a relative minimum point of the curve.

If $y = f(x)$ is differentiable on $a \leq x \leq b$ and if $f(x)$ has a relative maximum (minimum) value at $x = x_0$, where $a < x_0 < b$, then $f`(x_0) = 0$. For proof, see Prob. 18.

To find the relative maximum (minimum) values (hereinafter called maximum (minimum) values) of function $f(x)$ which, together with their first derivatives, are continuous:

FIRST DERIVATIVE TEST

1. Solve $f`(x) = 0$ for the critical values.
2. Locate the critical values on a number scale, thereby establishing a number of intervals.
3. Determine the sign of $f`(x)$ on each interval.
4. Let x *increase* through each critical value $x = x_0$; then
   $f(x)$ has a maximum value ($=f(x_0)$) if $f`(x)$ changes from + to -,
   $f(x)$ has a minimum value ($=f(x_0)$) if $f`(x)$ changes from - to +,
   $f(x)$ has neither a maximum nor a minimum value at $x =x_0$ if $f`(x)$ does not change sign.

A FUNCTION $y = f(x)$, necessarily less simple than those of Problems 2-5, may have a maximum or minimum value ($f(x_0)$) although $f`(x_0)$ does not exist. The value $x = x_0$ for which $f(x)$ is defined but $f`(x)$ does not exist will also be called critical vales for the function. They, together with the values for which $f`(x) = 0$ are to be used in determining the intervals of Step 2 above.

A final case in which $f(x_0)$ is a maximum (minimum) value although there is no interval $x_0 - 8 < x < x_0$ on which $f`(x)$ is positive (negative) and no interval $x_0 < x < x_0 + 8$ on which $f`(x)$ is negative (positive) will not be treated here.

DIRECTION OF BENDING. An arc of a curve $y = f(x)$ is called *concave upward* if, at each of its points, the arc lies above the tangent at the point. As x increases, $f`(x)$ either is of the same sign and increasing (as on the interval $b< x < s$ of Fig. 8-1) or changes sign from negative to positive (as on the interval $c < x < u$). In either case, the slope $f`(x)$ is increasing and $f``(x) >0$.

An arch of a curve $y = f(x)$ is called *concave downward* if, at each of its points, the arc lies below the tangent at the point. As x increases, $f`(x)$ either is of the same sign and decreasing (as on the interval $s < x < c$ of Fig. 8-1) or changes sign from positive to negative (as on the interval $a < x < b$). In either case, the slope $f`(x)$ is decreasing and $f``(x) < 0$.

A POINT OF INFLECTION is a point at which a curve is changing from concave upward to concave downward, or vice versa. In Fig. 8-1, the points of inflection are B, S, and C.

A curve $y = f(x)$ has one of its points $x = x_0$ as inflection point.

if $f``(x_0) =0$ or is not defined and

if $f``(x)$ changes sign as x increases through $x = x_0$.

The latter condition may be replaced by $f``(x_0) \neq 0$ *when* $f```(x_0)$ exist.

A SECOND TEST FOR MAXIMA AND MINIMA, SECOND DERIVATE TEST

1. Solve $f`(x) = 0$ for the critical values.
2. For a critical value $x = x_0$:

   $f(x)$ has a maximum value ($=f(x_0)$) if $f``(x_0) <0$

   $f(x)$ has a minimum value ($=f(x_0)$) if $f``(x_0) > 0$

In the later case, the first derivative method must be used.

6. RELATED RATES

RELATED RATES. If a variable x is a function of time t, the *time rate of change* of x is given by dx/dt.

When two or more variables, all functions of t, are related by an equation, the relation between their rates of change may be obtained by differentiating the equation with respect to t.

SOLVED PROBLEMS

1. Gas is escaping from a spherical balloon at the rate of 900 cm^3s^{-1}. How fast is the surface area shrinking when the radius is 360 cm?

   At time *t* the sphere has radius *r*, volume $V = \frac{4}{3}\pi r^3$, and surface $S = 4\pi r^2$.

   Then $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$, $\frac{dS/dt}{dV/dt} = \frac{2}{r}$, and

   $\frac{dS}{dt} = \frac{2}{r}\left(\frac{dV}{dt}\right) = \frac{2}{230}(-900) = -5cm^3s^{-1}$

2. Water is running out a conical funnel at the rate of 5 cm^3s^{-1}. If the radius of the base of the funnel is 10 cm and the altitude is 20 cm, find the rate at which the water level is dropping when it is 5 cm from the top.

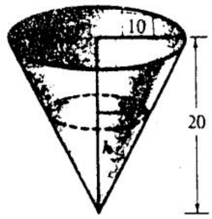
   

   Fig. 11-1

   Let *r* be radius and *h* the height of the surface of the water at time *t*, and *V* the volume of water in the cone.

   By similar triangles, $r/10 = h/20$ or $r = ½ h$. $V = 1/3\ \pi r^2 h = \frac{1}{12}\pi h^3$ and $dV/dt = ¼\ \pi h^2\ dh/dt$. When $dV/dt = -5$ and $h = 20 - 5 = 15$, then $dh/dt = -4/45\pi\ 5cm^3s^{-1}$.

3. Sand falling from a chute forms a conical pile whose altitude is always equal to 4/3 the radius of the base *(a)* How fast is the volume increasing when the radius of the base is 1 m and is increasing at the rate of 1/8 cm s^{-1}? *(b)* How fast is the radius increasing when it is 2 m and the volume is increasing at the rate of 10^4 cm^3s^{-1}?

   Let *r* = radius of base and *h* = height of pile at time *t*.

   Since $h = \frac{4}{3}r$, $v = \frac{1}{3}\pi r^2 h = \frac{4}{9}\pi r^2$, and $\frac{dV}{dt} = \frac{4}{3}\pi r^2 \frac{dr}{dt}$

   (a) When $r = 100$ and $\frac{dr}{dt} = \frac{1}{8}$, $\frac{dV}{dt} = \frac{5000\pi}{3} cm\ s^{-1}$

   (b) When $r = 200$ and $\frac{dV}{dt} = 10\,000$, $\frac{dr}{dt} = \frac{3}{16\pi} cm\ s^{-1}$

4. One ship *A* is sailing due to south at 24 km h^{-1} and a second ship *B*, 48 km south of *A*, is sailing due east at 18 km h^{-1} *(a)* At what rate are they approaching or separating at the end of 1 hr? *(b)* At the end of 2 hr? *(c)* When do they cease to approach each other and how far apart are they at that time?