MATHEMATICAL THINKING ACROSS MULTILATERAL CULTURE

By **Marsigit***)
Department of Mathematics Education, Faculty of Mathematics and Science, Yogyakarta State University

Abstract

Share the ideas and ways of mathematical thinking which are necessary for science, technology, economic growth and development of the APEC member economies, and develop the teaching approaches in mathematical thinking through Lesson Study among the APEC countries. To achieve these goals, since 2004 the APEC-International Conference on Innovative Teaching Mathematics through Lesson Study was held in Japan and Thailand.

Key Word: mathematical thinking, lesson study, multilateral culture

A. Background

The third[1] APEC Education Ministerial Meeting held on 29-30 April 2004 in Santiago, defined the priority areas for future network activities to stimulate learning in Mathematics and Science. Based on this priority, there were some activities of APEC project to encourage collaboration study on innovations for teaching and learning mathematics in different cultures. In 2004[2], Tsukuba University of Japan, Khon Kaen University of Thailand and Specialist Researchers from APEC Countries started to share the ideas and ways of mathematical thinking which are necessary for science, technology, economic growth and development of the APEC member economies, and develop the teaching approaches in mathematical thinking through Lesson Study among the APEC member economies.

The lesson study[3] that is attracting attention from around the world has actually derived from the education study in Japan since the days of normal school. In the field of arithmetic and mathematics, the collaboration between US and Japan since 1980s, resulted the schema to facilitate collaborative studies with various organizations around the world. Sponsored by APEC Economy countries, CRICED (Center for Research on International Cooperation in

**) Specialist Researcher on Mathematics Education in APEC Countries*

---

[1] Masami et al, I, 2006, *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (I): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*

[2] Ibid.

[3] Ibid.

Educational Development) of Tsukuba University, CRME (Center for Research in Mathematics Education) of Khon Kaen University, and the Specialists Researcher from the APEC economies has developed activities for four years with the focus on: mathematical thinking (2007), communication (2008), evaluation (2009), and generalization (2010).

The first three topics[4] were selected based on the three phases of the Lesson Study process: plan (for mathematical thinking), do (for communication) and see (for evaluation). Each year's results will become the basis for the following year's project. In the final year, generalization will be the theme, which will extend the implementation of Lesson Study to all subject areas.

In 2006 they have outlined the activities focusing on mathematical thinking, which is a necessary prerequisite for science, technology, economic growth and development. Using Lesson Study, the project aims to collaboratively: (1) share the ideas and ways of mathematical thinking which are necessary for science, technology, economic growth and development, and (2) develop the teaching approaches on mathematical thinking through Lesson Study among the APEC member economies. The Specialist Researchers from APEC economies contributed to develop lesson study by observing mathematics teaching in Japan and Thailand as well as in each of his/her country.

B. Mathematical Thinking as the Central Issues in Teaching Learning Mathematics Innovations

Mathematical thinking *(Ono Y, 2006*), is the basis for various types of thinking, and by learning mathematics students can learn the logical and rational mode of thinking. Also mathematics has a very wide range of applications including physics, statistics and economics. And in these various different fields mathematical thinking is employed. Also if we look at the curriculums in various countries, in any country we see, mathematics is taught from very young age. That is because all countries recognize the importance of mathematics. Following we will review some works of mathematics educationist from different context of culture in relation to the aspects of mathematical thinking

1. Australian context: *the works of Stacey Kaye*

Being able to use mathematical thinking in solving problems (Stacey, K. 2006), is one of the most the fundamental goals of teaching mathematics. It is an ultimate goal of teaching

[4] Ibid.

that students will be able to conduct mathematical investigations by themselves, and that they will be able to identify where the mathematics they have learned is applicable in real world situations. She indicated that mathematical thinking is important in three ways: as a goal of schooling, as a way of learning mathematics and for teaching mathematics. In this respect, mathematical thinking will support science, technology, economic life and development in an economy.

Accordingly mathematical thinking is a highly complex activity in which there are at least two process can be demonstrated: (1) specialising and generalising and (2) conjecturing and convincing. Since mathematical thinking is a process, it is probably best discussed through examples. There are many different 'windows' through which the mathematical thinking can be viewed. The organising committee for this conference (APEC, 2006) has provided a substantial discussion on this point. Stacey gives a review of how mathematical thinking is treated in curriculum documents in Australia.

In Australian context, Stacey K (2005) have found it helpful for teachers to consider that solving problems with mathematics requires a wide range of skills and abilities, including: (1) deep mathematical knowledge, (2) general reasoning abilities, (3) knowledge of heuristic strategies, (4) helpful beliefs and attitudes, (5) personal attributes such as confidence, persistence and organization, and (6) skills for communicating a solution. She then identified four fundamental processes, in two pairs, and showed how thinking mathematically very often proceeds by alternating between them:

- specialising – trying special cases, looking at examples
- generalising - looking for patterns and relationships
- conjecturing – predicting relationships and results
- convincing – finding and communicating reasons why something is true.

In her research, Stacey K (2005) found that considerable mathematical thinking on behalf of the teacher is necessary to provide a lesson that is rich in mathematical thinking for students. She uncovered that in mathematical thinking it needs for students to understand mathematical concepts and develop connections among concepts and the links between concepts and procedures. She also draws on important general mathematical principles such as[5] : (1) working systematically, (2) specialising – generalising: learning from examples by

[5] Stacey K, in Masami et al, I, 2006, *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (I): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*

looking for the general in the particular, (3) convincing: the need for justification, explanation and connections, and (4) the role of definitions in mathematics.

2. British context: *the works of David Tall*

David Tall (2006) argued that while teachers strive to improve performance on tests, there is a growing realization that practicing procedures to be able to perform them fluently is not sufficient to develop powerful mathematical thinking. Sometimes the detail that worked before may later prove to be inappropriate and cause difficulties. There are thus two important issues to address: taking account of ideas that students have met before that affect their current learning, and helping them to focus on essential ideas that become the basis of more subtle thinking.

David Tall intends to build on a theoretical framework for the long-term development of mathematical thinking from new-born child to adult which requires powerful ideas to be compressed into thinkable concepts that apply in new situations. He suggested that teachers need to act as mentors to rationalize the use of ideas that students have met before and to encourage knowledge into powerful ideas that can be linked together in coherent ways.

David Tall (ibid.), in the case of long-term learning of mathematical concepts, strived to explain how do students learn about mathematical concepts and how do they grow over the years to learn to think mathematically in sophisticated ways? He referred to Piaget that there are distinguished two fundamental modes of abstraction of properties from physical objects: *empirical abstraction* through teasing out the properties of the object itself, and *pseudo-empirical abstraction* through focusing on the actions on the objects, for instance, counting the number of objects in a collection as well as *reflective abstraction* focusing on operations on mental objects where the operation themselves become a focus of attention to form new concepts. Accordingly, he distinguishes two ways of building mathematical concept:

1) the first is from the exploration of a particular object whose properties he focus on and use first as a description – 'a triangle has three sides' – and then as a definition – 'a triangle is a figure consisting of three straight line segments joined end to end'.
2) the second arises from a focus on a sequence of actions and on organizing the sequence of actions as a mathematical procedure such as counting, addition, subtraction, multiplication, evaluation of an algebraic expression, computation of a function, differentiation, integration, and so on, with the compression into corresponding thinkable concepts such as number, sum, difference, product, expression, function, derivative, integral.

For a long-term mathematical thinking e.g. in geometry, David Tall (2006) emphasized Van Hiele's formulation consisting of building from perception of shapes, to description of their properties, practical constructions, definitions of figures that can be used for deductions, building to a coherent theory of Euclidean geometry. According to this formulation[6], the building of concepts from perception of, and actions on, physical objects and the growing sophistication towards definitions, deductions and formal theory is called the conceptual-embodied world of mathematical development. Two different forms of mathematical development[7], that interact at all levels, i.e. the **conceptual-embodied** (based on perception of and reflection on properties of objects) and the **proceptual-symbolic** that grows out of the embodied world through action (such as counting) and symbolization into thinkable concepts such as number, developing symbols that function both as processes to do and concepts to think about (called procepts); and the **axiomatic-formal** (based on formal definitions and proof) which reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretic definitions. are indicated in the following figure:

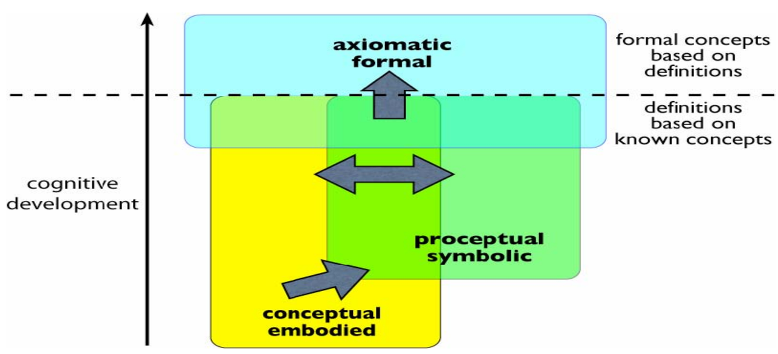

Figure: Interaction the level on mathematics thinking
Source: David Tall (2006)

3. Taiwaness Context: the works of Fou Lai Lin

Fou Lai Lin (2006) has developed a framework for designing conjecturing activity in mathematics thinking. He elaborated the entries of conjecturing and proved that conjecturing in mathematical thinking is a necessary process of problem solving, develops competency of proving and facilitates procedural operating. A conjecturing activity[8] may start with one of

[6] Tall D, in Masami et al, I, 2006, *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (I): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*
[7] Ibid.
[8] Lin F. L. in Masami et al, I, 2006, *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (I): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*

the three entries: a false statement, a true statement, and a conjecture of learners. Using students' misconception as starting point is an example, such as *A proceduralized refutation model* (PRM) (Lin & Wu, 2005) can be applied to design a conjecturing activity by substituting each students' misconception into the first item in the worksheet which follows student's activities step by step in the model.

Fou Lai Lin (2006) found that many teaching experiments show that high school students are able to notice the beauty of a certain formula. Students[9] are also convinced by applying the area formula with some *special/extreme cases* of triangles. Thinking in symmetry[10], degree of the expression and special/extreme cases composes a triad of mathematics thinking which can be generalized to make conjectures for formulae of geometry quantities. Refer to de Lange (1987) he agreed that mathematizing is an organizing and structuring activity according to which acquired knowledge and skills are used to discover unknown regularities, relations and structures. From Kilpatrick, Swafford, and Findell (2001) he noticed that mathematics proficiency consists five components: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

In his research, Fou Lai Lin (ibid.) strived to prove that conjecturing was able to enhance conceptual understanding. Using students' misconceptions as the starting statement in PRM, he investigated Freudenthal's claimed that conjecturing can enhance conceptual understanding both in prospective learning and in retrospective learning. He involved teachers to carry out their teaching exploration in which conjecturing is to facilitate procedural operating. Further, he found that conjecturing can develop competency of proving. Conjecturing and proving very often are discontinuous. In order to merge those two learning activities, learning strategy such as "constructing premise/conclusion" and "defining" are proved to be effective.

The ultimate results of his work suggest that conjecturing approach can drive innovation in mathematics teaching. He concluded that conjecturing activity encourages the students: (1) to construct extreme and paradigmatic examples, (2) to construct and test with different kind of examples, (3) to organize and classify all kinds of examples, (4) to realize structural features of supporting examples, (5) to find counter-examples when realizing a falsehood, (6) to experiment, (7) to self-regulate conceptually, (8) to evaluate one's own doing-thinking, (9) to formalize a mathematical statement, (10) to image /extrapolate/ explore a statement, and (11) to grasp fundamental principles of mathematics involves learners in *thinking and*

[9] Ibid.
[10] Ibid.

constructing actively.

4. Japanese Context: *the works of Katagiri*

Katagiri, S. (2004) insists that the most important ability that children need to gain at present and in future, as society, science, and technology advance dramatically, are not the abilities to correctly and quickly execute predetermined tasks and commands, but rather the abilities to determine themselves to what they should do or what they should charge themselves with doing. Of course, the ability[11] to correctly and quickly execute necessary mathematical problems is also necessary, but from now on, rather than adeptly to imitate the skilled methods or knowledge of others, the ability to come up with student's own ideas, no matter how small, and to execute student's own independence, preferable actions will be most important. Mathematical activities[12] cannot just be pulled out of a hat; they need to be carefully chosen so that children form concepts, develop skills, learn facts and acquire strategies for investigating and solving problems.

Mathematical thinking[13] has its diversity of simple knowledge or skills. It is evidence that mathematical thinking serves an important purpose in providing the ability to solve problems on one's own as described above, and this is not limited to this specific problem. Therefore, the cultivation of a number of these types of mathematical thinking should be the aim of mathematics teaching. Katagiri, S. (2004) lays out the followings as mathematical thinking related to mathematical method: inductive thinking, analogical thinking, deductive thinking, integrative thinking (including expansive thinking), developmental thinking, abstract thinking (thinking that abstracts, concretizes, idealizes, and thinking that clarifies conditions), thinking that simplifies, thinking that generalizes, thinking that specializes, thinking that symbolize, thinking that express with numbers, quantifies, and figures.

Teaching[14] should focus on mathematical thinking including mathematical method. Questions related to mathematical thinking and method must be posed based on a perspective of what kinds of questions to ask. Katagiri, S. (2004) indicates that quaestion must be created so that problem solving process elicits mathematical thinking and method. He lists question analysis designed to cultivate mathematical thinking as follows:

---

[11] Katagiri S. in Masami et al, I, 2006 *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (I): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*

[12] Ibid

[13] Ibid

[14] Ibid

a. Problem Formation and Comprehension

1) What is the same? What is shared? (Abstraction)
2) Clarify the meaning of the words and use them by oneself. (Abstraction)
3) What (conditions) are important? (Abstraction)
4) What types of situations are being considered? What types of situations are being proposed? (Idealization)
5) Use figures (numbers) for expression. (Diagramming, quantification)
6) Replace numbers with simpler numbers. (Simplification)
7) Simplify the conditions. (Simplification)
8) Give an example. (Concretization)

b. Establishing a Perspective

1) Is it possible to do this in the same way as something already known? (Analogy)
2) Will this turn out the same thing as something already known? (Analogy)
3) Consider special cases. (Specialization)

c. Executing Solutions

1) What kinds of rules seem to be involved? Try collecting data. (Induction)
2) Think based on what is known (what will be known). (Deduction)
3) What must be known before this can be said? (Deduction)
4) Consider a simple situation (using simple numbers or figures). (Simplification)
5) Hold the conditions constant. Consider the case with special conditions. (Specialization)
6) Can this be expressed as a figure? (Diagramming)
7) Can this be expressed with numbers? (Quantification)

d. Logical Organization

1) Why is this (always) correct? (Logical)
2) Can this be said more accurately? (Accuracy)

5. Singapore Context: *the works of Yeap Ban Har*

Yeap Ban Har (2006) illustrated that, in Singapore, education has an economic function. Education is perceived as preparing pupils to develop competencies that the future workforce needs to have. In particular[15], education is the platform to prepare pupils to become knowledge workers who are capable of innovative thinking and communicating such thinking. Thus, mathematical thinking[16] is a focus of the Singapore mathematics curriculum. Since 1992, the main aim of school mathematics has been to develop mathematical problem solving ability among pupils. The curriculum[17] was revised in 2001 and will be revised again in 2007 but the main aim remains the same.

[15] Yeap B. H. in Masami et al, I, 2006 *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (I): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*
[16] Ibid.
[17] Ibid

It was stated that, in 1997, the then Prime Minister of Singapore[18] announced that Singapore schools should help their pupils develop the ability to think. The Thinking Schools, Learning Nation initiative was started in 1997 for this purpose. Generic thinking skills such as classifying and comparing were taught to pupils. These thinking skills were also infused into key subjects including mathematics. Thinking skills are considered to be part of processes required in problem-solving efforts. In 2003, another initiative Innovation and Enterprise was introduced to encourage schools to develop good habits of mind or thinking habits among their pupils. Along with information technology and national education, thinking[19] is considered one of the key components of the education system.

Further, Yeap Ban Har (ibid.) indicated that pupils are expected to be able to engage in problem solving, routine as well as novel problem solving, in mathematics. This includes mathematical investigations. Mathematical thinking[20] is the process that pupils engage in when they solve mathematics problems. According to the curriculum framework, mathematical problem solving requires five inter-related components – skills, concepts, processes, attitude and metacognition. Pupils[21] are expected to possess mathematical skills and concepts. Skills include computation including mental computation and visualization. Key concepts[22] in elementary school include numerical, geometrical and algebraic concepts. Pupils are also expected to possess the ability to engage in processes such as reasoning, communicating, making connections, modeling, and using thinking skills and heuristics. This aspect[23] is the focus of Thinking Schools, Learning Nation. Pupils are expected to possess good problem-solving attitudes and habits as well as the ability to engage in metacognition. These aspects are the focus of Innovation and Enterprise.

Yeap Ban Har (ibid) stated that , the Singapore mathematics curriculum defines mathematical thinking as the orchestration of mathematical skills, concepts and processes to handle a situation which could be novel. This is reflected in the national examination. In the Primary School Leaving Examination (PSLE)[24] taken by pupils at the end of six years of primary education, about half of the maximum marks available for the mathematics test are from a section comprising thirteen problems. In this section, pupils must be able to show the method they used to solve the problems. He suggested that mathematical thinking as a

---

[18] Ibid
[19] Ibid
[20] Ibid
[21] Ibid
[22] Ibid
[23] Ibid
[24] Ibid

juxtaposition of mathematical competencies and generic competencies when pupils handle a mathematical situation such as mathematical problem solving. The mathematical competencies include visualization, patterning and number sense (Yeap, 2005). These mathematical competencies are referred to as 'big ideas'. These are the essence distilled from specific mathematical work that pupils engage in.

6. Malaysian Context: *the works of Lim Chap Sam*

In her preliminary research, Lim Chap Sam (2005) learned that for Malaysian context, it seem to highlight three major components of mathematical thinking: a) mathematical content / knowledge; b) mental operations; and c) predisposition. Referred to Beyer (1988), she indicated that to mathematical content/knowledge refers to the specific mathematics subject matter, mathematical concepts and ideas that one has acquired or learnt, while mental operations can be illustrated as cognitive activities that the mind needs to perform when thinking. Examples of predisposition include reasonableness, thinking alertness and open-mindedness, as well as beliefs and affects. Accordingly, she proposed that a working definition of mathematical thinking should include the following characteristics[25]: (1) it involves the manipulation of mental skills and strategies, (2) it is highly influenced by the tendencies, beliefs or attitudes of a thinker, (3) it shows the awareness and control of one's thinking such as meta-cognition, and (4) it is a knowledge−dependent activities. She then defined that mathematical thinking is a mental operation supported by mathematical knowledge and certain kind of predisposition, toward the attainment of solution to problem.

However, after a careful examination of the Malaysian school mathematics curriculum, both primary and secondary levels Lim Chap Sam (2006) indicated that The Mathematics curriculum for secondary school aims to develop individuals who are able to think mathematically and who can apply mathematical knowledge effectively and responsibly in solving problems and making decision. She found that all the three components of mathematical thinking are implicitly incorporated in both levels of Malaysian school mathematics curricula. For the primary mathematics curriculum[26], there is a higher emphasis on basic mathematical skills as compared to the problem solving skills and appreciation of mathematical values. In comparison[27], the emphasis is more on complex mathematical skills such as problem solving, decisions making, communication and extension of mathematical

[25] Sam L.C. in Masami et al, I, 2006 *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (I): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*
[26] Ibid.
[27] Ibid.

abstraction as well as positive attitudes toward mathematics rather then the basic mathematical skills for the secondary mathematics curriculum.

Lim Chap Sam's work indicated that there were some critical issues about mathematical thinking in Malaysia: (1) no clear understanding of mathematical thinking, (2) examination oriented culture and 'finish syllabus syndrome', (3) lack of appropriate assessment, (4) lack of resources and know-how in promoting mathematical thinking, (5) the role of technology in mathematical thinking. She proposed that to promote mathematical thinking it needs to equip and enhance mathematics teachers' understanding of mathematical thinking; a more explicit and comprehensive explanation of mathematical thinking will have to be stated in the school mathematics curriculum documents so that teachers can referred to these documents. Pre-service and in-service mathematics teachers[28] need to be made aware of the importance of mathematical thinking. They also need to be equipped with learning and to experience for themselves in mathematical thinking activities. These can be achieved by exposing mathematics teachers to various teaching strategies and activities that promote mathematical thinking. These ideas and activities can be imparted from time to time through workshops, seminars or conferences.

7. **Indonesian Context: *the works of Marsigit et.al***

Marsigit et al (2007) elaborated that the Decree of Sisdiknas No. 20 year 2003 insists that Indonesian Educational System should develop intelligence and skills of individuals, promote good conduct, patriotism, and social responsibility, should foster positive attitudes of self reliance and development. Improving the quality of teaching is one of the most important tasks in raising the standard of education in Indonesia. It was started in June 2006, based on the Ministerial Decree No 22, 23, 24 year 2006, Indonesian Government has implemented the new curriculum for primary and secondary education, called KTSP "School-Based Curriculum". This School-based curriculum combines two paradigms in which, one side stress on students competencies while on the other side concerns students' learning processes.

The School-Based Secondary Junior mathematics curriculum outlines that the aims of teaching learning of mathematics are as follows[29]: (1) to understand the concepts of mathematics, to explain the relationships among them and to apply them in solving the problems accurately and efficiently, (2) to develop thinking skills in learning patterns and

---

[28] Ibid.

[29] Direktorat SMP, 2006, "KTSP", Jakarta: Depdiknas

characteristics of mathematics, to manipulate them in order to generalize, to prove and to explain ideas and mathematics propositions, (3) to develop problem solving skills which cover understanding the problems, outlining mathmatical models, solving them and estimating the outcomes, (4) to communicate mathematics ideas using symbols, tables, diagrams and other media, and (5) to develop appreciations of the use of mathematics in daily lifes, curiosity, consideration, and to encourage willingness and self-confidence.in learning mathematics.

According to Marsigit et al (2007), for Indonesian context, the aim of mathematics education from now on is still urgently to promote mathematical thinking and to take it into actions. Accordingly, these lead to suggest that it needs to conduct classroom-based research to investigate the necessary driving factors towards students' ability to develop mathematical thinking. Marsigit's work indicated that mathematics would have to be applied to natural situations, any where real problems appear, and to solve them, it is necessary to use the mathematical method. The knowledge[30], skills, and mathematical methods are the foundation to achieve the knowledge on science, information, and other learning areas in which mathematical concepts are central; and to apply mathematics in the real-life situations. This study uncovered that teacher has important role to encourage their students to develop mathematical methods.

On the study of uncovering students' developing mathematical thinking in learning the total area of a right circular cylinder and sphere and also the volume of a right circular cone in the 8th grade of Junior High School, Marsigit et al (2007) found that the students performed *mathematical thinking* when they found difficulties or when they were asked by the teacher. Most of the students reflected that they paid attention on the perfect of the *Concrete Model* of geometrical shape. However, their consideration on the perfect form of the models did not indicate that they performed *mathematical idealization* as one of mathematical method. Marsigit (ibid) also found that, one aspect of mathematical method i.e. s*implifications* happened when the students perceived that the concept of *right circular cone* is similar to the concept of triangle or circle. In this case, they *simplified* the concepts through manipulation of *Concrete Models.* They also performed simplification when they broke down the formula to solve the problems. They mostly simplified the concepts when they had got some questions from the teacher; or, when they worked in group.

[30] Marsigit et al in Masami et al, I, 2007 *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (II): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*

Ultimately, in this work, Marsigit et al (ibid) found that the students developed *inductive thinking* when they uncovered that the height of right circular cylinder is equal to the width of its rectangle; and the circumference of the circle is equal to the length of rectangle. They continued to perform *inductive thinking*[31] until they found the formula of the lateral area of right circular cylinder; the formula of sphere, and the formula of the volume of *Right Circular Cone.* Students' *schema of inductive thinking* covers: (1) attempting to gather a certain amount of data, (2) working to discover rules or properties in common between these data, (3) inferring that the set that includes that data (the entire domain of variables) is comprised of the discovered rules and properties, and (4) confirming the correctness of the inferred generality with new data.

In the latest Lesson Study, Marsigit et al (2007) had sought to uncover the picture in which the teacher strived to promote mathematical thinking in learning the total area of a right circular cylinder and sphere as well as the volume of a right circular cone. S*tudents' mathematical thinking* can be traced through the schema of teaching learning activities as follows:

1. Problem Formation and Comprehension were emerged when the students:
   a. observed given model of right circular cylinder, observed given model of Sphere, and observed given model of right circular cone
   b. identified the components of the right circular cylinder, sphere, and right circular cone
   c. defined the concept of right circular cylinder, sphere, and right circular cone
   d. got questions and notices from teacher to search the concepts
2. Establishing a Perspective were emerged when the students:
   a. employed concrete model to search the total area of right circular cylinder, the area of sphere and the volume of right circular cone
   b. learned that the height of right circular cylinder is equal to the width of its rectangle; and the circumference of the circle is equal to the length of rectangle
   c. learned the teacher's guide to understand the procedures how to search the volume of right circular cone
   d. broke-down the model of right circular cylinder into its components
3. Executing Solutions were emerged when the students:
   a. tried to find out the lateral area of right circular cylinder
   b. tried to find out the total area of right circular cylinder

[31] Ibid.

c. tried to find out the area of sphere
d. collected the data of the measurement of the volume of cone in comparison with the volume of cylinder

C. Discussion

In Australia, if students are to become good mathematical thinkers, then mathematicalthinking needs to be a prominent part of their education. In addition, however students[32] who have an understanding of the components of mathematical thinking will be able to use these abilities independently to make sense of mathematics that they are learning. For example, if they do not understand what a question is asking, they should decide themselves to try an example (specialise) to see what happens, and if they are oriented to constructing convincing arguments, then they can learn from reasons rather than rules. Experiences like the exploration above, at an appropriate level build these dispositions.

As indicated by Stacey, K, for Australian context, mathematical thinking is not only important for solving mathematical problems and for learning mathematics. A teacher[33] requires mathematical thinking for analysing subject matter planning lessons for a specified aim and anticipating students' responses. These are indeed key places where mathematical thinking is required. Mathematical thinking[34] is not just in planning lessons and curricula; it makes a difference to every minute of the lesson. In the case of teaching mathematics, the solver has to bring together expertise in both mathematics and in general pedagogy, and combine these two domains of knowledge together to solve the problem, whether it be to analyse subj ect matter, to create a plan for a good lesson, or on a minute-by-minute basis to respond to students in a mathematically productive way. If teachers are to encourage mathematical thinking in students, then they need to engage in mathematical thinking throughout the lesson themselves.

For British context, David Tall (2006) lead to a long-term view of mathematics thinking, building on the genetic capabilities of the learner and the successive learning experiences over a life-time: (1) the child is born with generic capabilities *set-before* in the genetic structure, (2) current cognitive development builds on experiences that were *met-before,* (3) this occurs through long-term potentiation of neuronal connections which strengthens successful links and suppresses others, (4) actions are coordinated as

[32] Stacey K, in Masami et al, I, 2006, *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (I): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*
[33] Ibid
[34] Ibid

(procedural) *action-schemas*, (5) ideas are compressed into *thinkable concepts* using language & symbolism, (6) thinkable concepts are built into wider (conceptual) *knowledge schemas,* (7) mathematical thinking builds cognitively through *embodiment, symbolism* and, later, *formal proof,* each developing in sophistication over time, (8) success in mathematical thinking depends on the effect of met-befores, the compression to rich thinkable concepts, and the building of successive levels of sophistication that is both powerful and simple.

David Tall indicated that various studies carried out by doctoral students at Warwick University in countries around the world reveal a widespread goal of 'raising standards' in mathematics learning, which are tested by tests that *could* promote conceptual long-term learning, but in practice, often produce short-term procedural learning that is may be less successful in developing long-term flexibility in understanding and solving non-routine problems. Looking at the total picture of long-term learning, what emerges is the absolute necessity of the teacher helping the student to construct thinkable concepts that not only enable students to solve current problems, but also to move on to greater sophistication. In a given situation, the learning of efficient procedures to *do* mathematics is an important part of learning, but in the long-term, it is essential to compress knowledge into thinkable concepts that will work in more sophisticated ways. This can be done by building on embodied experiences that can give insightful meanings suitable for initial learning but may include met-befores that can hinder future sophistication. Here it is essential to focus on the development of flexible thinking with the symbolism that compresses processes that can be used to solve mathematical problems into procepts that can be used to *think* about mathematics.

From the view point of Lesson Study, mathematical thinking should be developed through lessons. Usually, mathematical thinking is defined by the curriculum and embedded in the aim of each lesson. Thus, curriculum documents of each economy would be the clearest resources for analysing what mathematical thinking is in each economy (Masami Isoda, 2006). Accordingly, in the Japanese curriculum, mathematical thinking has been defined for clarifying the quality of activity since 1951 for secondary school and since 1953 for elementary and middle school. In Japanese curriculum documents, mathematical thinking is defined with mathematizing activity, and it has three components to be taught: the ability of 'see as', 'ways of thinking', and 'appreciation of its significance'. In Japan, there are four categories of evaluation standards: attitude, mathematical thinking, representation, and understanding. Each category is related to the others.

In Japan, mathematical thinking [35]is based on mathematical attitude, is carried out with mathematical representation and is necessary for understanding. The order of these four categories resembles the process of thinking, but it is not specific to mathematics because similar conditions exist in other academic subjects. The Japanese Ministry of Education[36] recommended that teachers have decision making authority for teaching a lesson based on the observation conditions developed from these four categories. In lesson planning[37] during the first part of Lesson Study, teachers analyse subject matter and anticipate students' responses. In this process, teachers plan the lesson keeping in mind the four categories. Thus, the Ministry recommended that teachers describe these four categories with specific mathematical conceptions which should appear in a specific lesson.

In Indonesia, as it happened also in Malaysia, it's pop-up like a jack. The examination oriented culture[38] is still prevalent in Indonesian and Malaysian schools, in spite of the government's effort to "humanize" the public assessment system recently. Examination results, especially the public examination result remain to be used as a yard stick or accountability of school performance. It[39] is also common for school principals to use students' performance as appraisal to assess teachers' teaching performance. Under[40] the pressure of achieving excellent examination results, it is not surprising to observe that most teachers tended to teach to test. They[41] were more anxious to finish the syllabus so as to answer to the expectation of the school principal and parents, regardless of students' understanding and learning. This kind of "finish the syllabus syndrome" often render teachers no choice but to use procedural teaching that is a fast and direct way of information/knowledge transfer. Many teachers[42] stress on "drill and practice" so that students are familiar with the style of examination questions. Students are taught to master the answering techniques, instead of executing mathematics thinking skills and strategies to solve the problems.

---

[35] Masami et al, I, 2006, *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (I): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*

[36] Ibid.

[37] Ibid.

[38] Sam L.C. in Masami et al, I, 2006 *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (I): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*

[39] Ibid.

[40] Ibid.

[41] Ibid.

[42] Ibid.

Beside the lack of clear understanding[43] about mathematical thinking, teachers generally do not receive enough support from their school, especially in terms of teaching and learning materials, references and professional development training. Furthermore, most teachers experienced their school mathematics learning through procedural approach. Many of them tended to teach as they were taught. Hence, many teachers still lack the know-how and resources to incorporate mathematical thinking activity in their mathematics lessons. They[44] need extra time and effort in preparation, while time is the biggest constraint in view of the examination oriented culture and heavy workload of teachers. Consequently, this discourages many teachers from integrating mathematical thinking activity in their lessons.

D. Conclusion

Mathematical thinking has meant many things for many educationists. There are some features in which we can promote mathematics thinking such as follows:

1. The first feature is reorganization through mathematization by reflective thinking.
2. The second feature is acquisition and using mathematical concept on ideal world
3. The third feature is learning how to learn, develop and use mathematics in the previous two types of learning.
4. Share the ideas and ways of mathematical thinking which are necessary for science, technology, economic growth and development, and
5. Develop the teaching approaches on mathematical thinking through Lesson Study
6. Develop networks for sharing ideas on performing mathematics thinking at national, regional or international level.

Reference:

Bonomo, M.F.C (2006), *Mathematical Thinking Like Angular Stone In The Understanding Of Real World Phenomena,* in Progress report of the APEC project: "Colaborative Studies on Innovations for Teaching and Learning Mathematics in Diferent Cultures (II) - Lesson Study focusing on Mathematical Thinking -", Tokyo: CRICED, University of Tsukuba.

Isoda, M. (2006). *First Announcement : APEC-Tsukuba International Conference on InnovativeTeaching Mathematics Through Lesson Study (II) – Focussing on Mathematical Thinking-December 2-7, 2006,* Tokyo & Sapporo, Japan

[43] Ibid.
[44] Ibid.

Lange, J. de (2006). *Mathematical Literacy for Living From OECD-PISA Perspective*, Tokyo: Simposium on International Cooperation

Masami et al, I, 2006, *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (I): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*

Marsigit, (2006), *Lesson Study: Promoting Student Thinking On TheConcept Of Least Common Multiple (LCM) Through Realistic Approach In The 4th Grade Of Primary Mathematics Teaching,* in Progress report of the APEC project: "Colaborative Studies on Innovations for Teaching and Learning Mathematics in Diferent Cultures (II) – Lesson Study focusing on Mathematical Thinking -", Tokyo: CRICED, University of Tsukuba.

Shikgeo Katagiri (2004)., *Mathematical Thinking and How to Teach It*. in Progress report of the APEC project: "Colaborative Studies on Innovations for Teaching and Learning Mathematics in Diferent Cultures (II) – Lesson Study focusing on Mathematical Thinking -", Tokyo: CRICED, University of Tsukuba.

Stacey K, (2006), *What Is Mathematical Thinking And Why Is It Important?* in Progress report of the APEC project: "Colaborative Studies on Innovations for Teaching and Learning Mathematics in Diferent Cultures (II) – Lesson Study focusing on Mathematical Thinking -", Tokyo: CRICED, University of Tsukuba.

Stacey K, in Masami et al, I, 2006, *"Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures (I): Lesson Study on Mathematical Thinking", Tsukuba University: CRICED*

Tall D. (2006), *Encouraging Mathematical Thinking That Has Both Power And Simplicity* in Progress report of the APEC project: "Colaborative Studies on Innovations for Teaching and Learning Mathematics in Diferent Cultures (II) – Lesson Study focusing on Mathematical Thinking -", Tokyo: CRICED, University of Tsukuba.